

A GENERALIZATION OF FINITE-DIMENSIONAL GORENSTEIN ALGEBRAS

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ABSTRACT. We introduce and study abelian categories equipped with a comonad and a Nakayama functor relative to the comonad. These categories generalize important features of the module category of a finite-dimensional algebra from the viewpoint of Gorenstein homological algebra. In particular, we define Gorenstein flat and Gorenstein injective objects relative to the comonad in such categories. We also develop a theory of Gorenstein comonads and obtain analogues of several classical results for Iwanaga-Gorenstein algebras. One of our main examples is the module category $\Lambda\text{-Mod}$ of a k -algebra Λ , where k is a commutative ring and Λ is finitely generated projective as a k -module. In this case, we obtain a new result stating that if the projective dimension of $\text{Hom}_k(\Lambda, k)$ is finite as a right and left Λ -module, then these two dimensions coincide. Finally, we also state a conjecture generalizing the Gorenstein symmetry conjecture for finite-dimensional algebras.

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Date: November 8, 2016.

2010 Mathematics Subject Classification. 18E10, 16E65, 16D90;

Key words and phrases. Abelian category; comonad; Gorenstein homological algebra; Gorenstein ring; homological algebra .

The author thanks Gustavo Jasso, Julian Külshammer, Rosanna Laking, and Jan Schröer for helpful comments on a previous version of this paper. A special thanks goes to Julian Külshammer for suggesting a simplified proof of Lemma 4.3.3 using Kleisli categories. The work was made possible by the funding provided by the *Bonn International Graduate School in Mathematics*.

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1. INTRODUCTION

1.1. Introduction. A left and right Noetherian ring Λ is *Iwanaga-Gorenstein* if it has finite injective dimension as a left and right module over itself. This is an important and well studied class of rings. They generalize algebras of finite global dimension, but are restrictive enough to still retain interesting properties. A classical result by Iwanaga [Iw] states that

$$\text{inj. dim } \Lambda_{\Lambda} = \text{inj. dim } {}_{\Lambda}\Lambda$$

when Λ is Iwanaga-Gorenstein. If this dimension is n we say that Λ is n -Gorenstein. In this case, a module has finite projective dimension if and only if it has finite injective dimension, and the maximal finite projective and finite injective dimension is the injective dimension of Λ . Iwanaga-Gorenstein rings play a central role in *Gorenstein homological algebra*, see [Z, C1, H, EJ]. The central actors in this theory are the *Gorenstein projective modules* and *Gorenstein injective modules*. For an Iwanaga-Gorenstein ring these give rise to cotorsion pairs [Ho]. The category of Gorenstein projective modules is a Frobenius category, and a classical result by Buchweitz [Bu] states that for an Iwanaga-Gorenstein algebra Λ the stable category of finitely presented Gorenstein projective modules is equivalent to the singularity category

$$D_{\text{sg}}(\Lambda) = D^b(\Lambda\text{-mod})/K^b(\text{proj}(\Lambda\text{-mod})).$$

Recently, the properties of $\Lambda\text{-Mod}$ when Λ is Iwanaga-Gorenstein have been generalized to Grothendieck categories [EEG], and this has been used to show some universal coefficient theorems for triangulated categories [DSS].

There are many interesting examples of Iwanaga-Gorenstein algebras which are finite-dimensional, see for example [GLS, GR, KR]. In this case the Gorenstein projective modules coincide with the Gorenstein flat modules, and many of the statements for Iwanaga-Gorenstein rings, Gorenstein projective modules, and Gorenstein injective modules can be reformulated. We spell out some of the most important results. Assume Λ_{fd} is a finite-dimensional algebra over a field k , and let $D = \text{Hom}_k(-, k)$ be the dual. Let $\Lambda_{\text{fd}}\text{-mod}$ be the category of finite-dimensional Λ_{fd} -modules. The following hold:

- (G1) A module $G \in \Lambda_{\text{fd}}\text{-mod}$ is Gorenstein projective (=Gorenstein flat) if and only if there exists an exact complex

$$Q_{\bullet} = \cdots \rightarrow Q_{-1} \rightarrow Q_0 \rightarrow Q_1 \rightarrow \cdots$$

with projective components, such that the complex $D(\Lambda_{\text{fd}}) \otimes_{\Lambda_{\text{fd}}} Q_{\bullet}$ is exact, and with $G = \text{im}(Q_{-1} \rightarrow Q_0)$ [C1, Lemma 2.2.9]. Dually, a module $G \in \Lambda_{\text{fd}}\text{-mod}$ is Gorenstein injective if and only if there exist an exact complex

$$J_{\bullet} = \cdots \rightarrow J_{-1} \rightarrow J_0 \rightarrow J_1 \rightarrow \cdots$$

with injective components, such that the complex

$\text{Hom}_{\Lambda_{\text{fd}}\text{-mod}}(D(\Lambda_{\text{fd}}), J_{\bullet})$ is exact, and with $G = \text{im}(J_{-1} \rightarrow J_0)$, see [C1, Section 2].

- (G2) The category $\mathcal{GP}(\Lambda_{\text{fd}}\text{-mod})$ of Gorenstein projective modules is resolving, and the category $\mathcal{GI}(\Lambda_{\text{fd}}\text{-mod})$ of Gorenstein injective modules is coresolving, see [C1, Proposition 2.1.7]. Hence, we can define the dimensions $\text{res. dim}_{\mathcal{GP}(\Lambda_{\text{fd}}\text{-mod})} \Lambda_{\text{fd}}\text{-mod}$ and $\text{cores. dim}_{\mathcal{GI}(\Lambda_{\text{fd}}\text{-mod})} \Lambda_{\text{fd}}\text{-mod}$ of $\Lambda_{\text{fd}}\text{-mod}$ with respect to these categories, see [S, Section 2].

- (G3) The algebra Λ_{fd} is Iwanaga-Gorenstein if and only if

$$\text{proj. dim}_{\Lambda_{\text{fd}}} D(\Lambda_{\text{fd}}) < \infty \quad \text{and} \quad \text{proj. dim}_{D(\Lambda_{\text{fd}})\Lambda_{\text{fd}}} D(\Lambda_{\text{fd}}) < \infty.$$

- (G4) If Λ_{fd} is Iwanaga-Gorenstein, then

$$\mathcal{GP}(\Lambda_{\text{fd}}\text{-mod}) = \{M \in \Lambda_{\text{fd}}\text{-mod} \mid \text{Tor}_i^{\Lambda_{\text{fd}}}(D(\Lambda_{\text{fd}}), M) = 0 \ \forall i > 0\}$$

$$\mathcal{GI}(\Lambda_{\text{fd}}\text{-mod}) = \{M \in \Lambda_{\text{fd}}\text{-mod} \mid \text{Ext}_{\Lambda_{\text{fd}}\text{-mod}}^i(D(\Lambda_{\text{fd}}), M) = 0 \ \forall i > 0\}$$

see [C1, Theorem 2.3.3].

- (G5) If Λ_{fd} is Iwanaga-Gorenstein then

$$\text{proj. dim}_{\Lambda_{\text{fd}}} D(\Lambda_{\text{fd}}) = \text{proj. dim}_{D(\Lambda_{\text{fd}})\Lambda_{\text{fd}}} D(\Lambda_{\text{fd}}).$$

- (G6) The following statements are equivalent, see [C1, Theorem 2.3.3]:

- Λ_{fd} is n -Gorenstein;
- $\text{res. dim}_{\mathcal{GP}(\Lambda_{\text{fd}}\text{-mod})} \Lambda_{\text{fd}}\text{-mod} = n$;
- $\text{cores. dim}_{\mathcal{GI}(\Lambda_{\text{fd}}\text{-mod})} \Lambda_{\text{fd}}\text{-mod} = n$.

On the other hand, we also have a *comonad* (see Definition 2.2.1) $P_{\text{fd}} = (P_{\text{fd}}, \epsilon, \Delta)$ on $\Lambda_{\text{fd}}\text{-mod}$ where

$$P_{\text{fd}} := (\Lambda_{\text{fd}} \otimes_k -) \circ \text{res}_k^{\Lambda_{\text{fd}}} : \Lambda_{\text{fd}}\text{-mod} \rightarrow \Lambda_{\text{fd}}\text{-mod}$$

and $\text{res}_k^{\Lambda_{\text{fd}}} : \Lambda_{\text{fd}}\text{-mod} \rightarrow \text{mod } k$ is the restriction functor, see 2.4.11. This is the main ingredient that we generalize; we show that a similar theory can be developed for any abelian category \mathcal{A} equipped with a comonad P and a Nakayama functor relative to P .

We now give an account of our constructions and results. Fix an abelian category \mathcal{A} , a functor $P: \mathcal{A} \rightarrow \mathcal{A}$, and a comonad $P = (P, \epsilon, \Delta)$ on \mathcal{A} . Note that this is equivalent to fixing a monad on \mathcal{A}^{op} . In fact, one can develop a similar theory of monads instead of comonads. The reason we work with comonads is that this corresponds to working with projective objects instead of injective objects, which we prefer.

Definition 1.1.1 (See Definition 4.1.1). Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} . A *Nakayama functor* relative to P is a functor $\nu: \mathcal{A} \rightarrow \mathcal{A}$ with a right adjoint ν^- , and satisfying the following:

- (i) $\nu \circ P$ is right adjoint to P ;
- (ii) The unit $\lambda: 1_{\mathcal{A}} \rightarrow \nu^- \circ \nu$ induces an isomorphism on objects of the form $P(A)$ for $A \in \mathcal{A}$.

Here P is *generating* (see Definition 2.4.1) if $\epsilon: P \rightarrow 1_{\mathcal{A}}$ is an epimorphism. We show that a Nakayama functor relative to P is unique if it exists, see Theorem 4.3.9. Hence, having a Nakayama functor should be thought of as a property of the comonad P . In the example above the Nakayama functor relative to P_{fd} is just the classical Nakayama functor

$$\nu_{\text{fd}} = D(\Lambda_{\text{fd}}) \otimes_{\Lambda} -: \Lambda_{\text{fd}}\text{-mod} \rightarrow \Lambda_{\text{fd}}\text{-mod}.$$

Example 1.1.2. Let k be a commutative ring, and let Λ be a k -algebra which is finitely generated projective as a k -module. Orders over complete regular local rings [I2] are examples of such algebras. Let $P_{\Lambda\text{-Mod}} = (P_{\Lambda\text{-Mod}}, \epsilon, \Delta)$ be the comonad on $\Lambda\text{-Mod}$ induced from the functor

$$P_{\Lambda\text{-Mod}} := (\Lambda \otimes_k -) \circ \text{res}_k^{\Lambda}: \Lambda\text{-Mod} \rightarrow \Lambda\text{-Mod}.$$

see Example 2.4.11. Then $P_{\Lambda\text{-Mod}}$ is generating and the functor

$$\nu_{\Lambda\text{-Mod}} = D(\Lambda) \otimes_{\Lambda} -: \Lambda\text{-Mod} \rightarrow \Lambda\text{-Mod}$$

is a Nakayama functor relative to $P_{\Lambda\text{-Mod}}$, where $D := \text{Hom}_k(-, k)$ is the algebraic dual. As far as we know, the results we obtain are new even in this example.

More generally, for any commutative ring k the module category $\mathcal{C}\text{-Mod}$ of a small, k -linear, locally bounded and Hom-finite category \mathcal{C} has a comonad and a Nakayama functor relative to the comonad, see Example 4.1.3. Such categories are for example studied in [DSS]. In an upcoming paper we will show that for such a \mathcal{C} the functor category $\mathcal{A} = \mathcal{B}^{\mathcal{C}}$ of an abelian category \mathcal{B} has a comonad with a Nakayama functor relative to the comonad. We also expect that there exist graded versions of these examples.

Definition 1.1.3 (See Definition 3.1.2 and Lemma 4.2.1). Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} with Nakayama functor ν relative to P . An object $G \in \mathcal{A}$ is *Gorenstein P -flat* if there exists an exact sequence

$$Q_{\bullet} = \cdots \rightarrow Q_{-1} \rightarrow Q_0 \rightarrow Q_1 \rightarrow \cdots$$

in \mathcal{A} , where $Q_i = P(A_i)$ for $A_i \in \mathcal{A}$, such that the complex $\nu(Q_\bullet)$ is exact, and with $Z^0(Q_\bullet) = G$.

A module $G \in \Lambda\text{-Mod}$ is Gorenstein $P_{\Lambda\text{-Mod}}$ -flat if there exists an exact sequence

$$Q_\bullet = \cdots \rightarrow Q_{-1} \rightarrow Q_0 \rightarrow Q_1 \rightarrow \cdots$$

in $\Lambda\text{-Mod}$, where $Q_i = \Lambda \otimes_k M_i$ for $M_i \in \Lambda\text{-Mod}$, such that the complex $D(\Lambda) \otimes_\Lambda Q_\bullet$ is exact, and with $Z^0(Q_\bullet) = G$. Also, it follows from (G1) that the Gorenstein P_{fd} -flat modules in $\Lambda_{\text{fd}}\text{-mod}$ are precisely the finite-dimensional Gorenstein projective Λ_{fd} -modules.

In the following we let $I := \nu \circ P: \mathcal{A} \rightarrow \mathcal{A}$ and $\mathsf{l} = (I, \eta, \mu)$ be the induced monad on \mathcal{A} , see Proposition 2.2.6.

Definition 1.1.4 (See Definition 3.1.7 and Lemma 4.2.1). Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} with Nakayama functor ν relative to P . An object $G \in \mathcal{A}$ is *Gorenstein l -injective* if there exists an exact sequence

$$J_\bullet = \cdots \rightarrow J_{-1} \rightarrow J_0 \rightarrow J_1 \rightarrow \cdots$$

in \mathcal{A} , where $J_i = I(A_i)$ for $A_i \in \mathcal{A}$, such that the complex $\nu^-(J_\bullet)$ is exact, and with $Z^0(J_\bullet) = G$.

In the example above the right adjoint to $P_{\Lambda\text{-Mod}}$ is

$$I_{\Lambda\text{-Mod}} := \text{Hom}_k(\Lambda, -) \circ \text{res}_k^\Lambda: \Lambda\text{-Mod} \rightarrow \Lambda\text{-Mod}.$$

It follows that a module $G \in \Lambda\text{-Mod}$ is Gorenstein $\mathsf{l}_{\Lambda\text{-Mod}}$ -injective if there exists an exact sequence

$$J_\bullet = \cdots \rightarrow J_{-1} \rightarrow J_0 \rightarrow J_1 \rightarrow \cdots$$

in $\Lambda\text{-Mod}$, where $J_i = \text{Hom}_k(\Lambda, M_i)$ for $M_i \in \Lambda\text{-Mod}$, such that the complex $\text{Hom}_{\Lambda\text{-Mod}}(D(\Lambda), J_\bullet)$ is exact, and with $Z^0(J_\bullet) = G$. It follows from (G1) that the Gorenstein l_{fd} -injective modules in $\Lambda_{\text{fd}}\text{-mod}$ are precisely the finite-dimensional Gorenstein injective Λ_{fd} -modules.

Theorem 1.1.5. *Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} with a Nakayama functor relative to P .*

- (i) *The category $\mathcal{G}_P \text{flat}(\mathcal{A})$ of Gorenstein P -flat modules forms a resolving subcategory of \mathcal{A} (Corollary 3.2.17);*
- (ii) *The category $\mathcal{G}_{\mathsf{l}} \text{inj}(\mathcal{A})$ of Gorenstein l -injective modules forms a coresolving subcategory of \mathcal{A} (Theorem 3.2.18).*

It is not necessary to have a comonad with Nakayama functor for this theorem to hold. In fact, the definition of Gorenstein P -flat modules and Theorem 1.1.5 part (i) holds for any comonad P which *accommodates Gorenstein objects*, see Definition 3.1.1. Dually, the definition of Gorenstein l -injective modules and Theorem 1.1.5 part (ii) holds for any monad l which *accommodates Gorenstein objects*, see Definition 3.1.6. In 3.1.5 we give an example of a comonad which accommodates Gorenstein objects, but which doesn't necessarily have a Nakayama functor.

Definition 1.1.6 (See Definition 5.1.2). Let $\mathbf{P} = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} with a Nakayama functor ν relative to \mathbf{P} . We say that \mathbf{P} is *Gorenstein* if there exists an $n \in \mathbb{N}$ such that $H_i(\nu; A) = 0$ and $H^i(\nu^-; A) = 0$ for all $A \in \mathcal{A}$ and $i \geq n$.

Here $H_i(\nu; A)$ and $H^i(\nu^-; A)$ denotes the comonadic and monadic homology with respect to \mathbf{P} and \mathbf{I} , see Section 2.3. If \mathcal{A} has enough projectives, then $H_i(\nu; -)$ coincides with the i th left derived functor of ν , see 2.4.10. Dually, if \mathcal{A} has enough injectives, then $H^i(\nu^-)$ coincides with the i th right derived functor of ν^- . Note that by (G3) the comonad \mathbf{P}_{fd} is Gorenstein if and only if Λ_{fd} is an Iwanaga-Gorenstein algebra. More generally, it follows from Lemma 5.2.1 that the comonad $\mathbf{P}_{\Lambda\text{-Mod}}$ is Gorenstein if and only if

$$\text{proj. dim } {}_{\Lambda}D(\Lambda) < \infty \quad \text{and} \quad \text{proj. dim } D(\Lambda)_{\Lambda} < \infty.$$

For an Iwanaga-Gorenstein ring the category of Gorenstein projective modules is the left perpendicular of the projective modules. We get an analogous result for comonads which are Gorenstein.

Theorem 1.1.7 (Theorem 5.1.3). *Let \mathbf{P} be a generating comonad on \mathcal{A} with a Nakayama functor ν relative to \mathbf{P} . If \mathbf{P} is Gorenstein, then*

$$\begin{aligned} \mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A}) &= \{A \in \mathcal{A} \mid H_i(\nu; A) = 0 \ \forall i \in \mathbb{N}\}, \\ \mathcal{G}_{\mathbf{I}} \text{ inj}(\mathcal{A}) &= \{A \in \mathcal{A} \mid H^i(\nu^-; A) = 0 \ \forall i \in \mathbb{N}\}. \end{aligned}$$

For the comonad \mathbf{P}_{fd} this theorem is equivalent to (G4). For the comonad $\mathbf{P}_{\Lambda\text{-Mod}}$ it translates to

$$\begin{aligned} \mathcal{G}_{\mathbf{P}_{\Lambda\text{-Mod}}} \text{ flat}(\Lambda\text{-Mod}) &= \{M \in \Lambda\text{-Mod} \mid \text{Tor}_i^{\Lambda}(D(\Lambda), M) = 0 \ \forall i \in \mathbb{N}\}, \\ \mathcal{G}_{\mathbf{I}_{\Lambda\text{-Mod}}} \text{ inj}(\Lambda\text{-Mod}) &= \{M \in \Lambda\text{-Mod} \mid \text{Ext}_{\Lambda\text{-Mod}}^i(D(\Lambda), M) = 0 \ \forall i \in \mathbb{N}\} \end{aligned}$$

when $\mathbf{P}_{\Lambda\text{-Mod}}$ is Gorenstein.

The following theorem is an analogue of Iwanaga's result on the injective dimension of Iwanaga-Gorenstein algebras.

Theorem 1.1.8 (See Theorem 5.1.7). *Let \mathbf{P} be a generating comonad on \mathcal{A} with a Nakayama functor ν relative to \mathbf{P} . If \mathbf{P} is Gorenstein, then the following numbers coincide:*

- (i) *The smallest integer n_1 such that $H_i(\nu; A) = 0$ for all $i \geq n_1$ and $A \in \mathcal{A}$;*
- (ii) *The smallest integer n_2 such that $H^i(\nu^-; A) = 0$ for all $i \geq n_2$ and $A \in \mathcal{A}$.*

We say that \mathbf{P} is n -Gorenstein if this common number is n . Note that the comonad \mathbf{P}_{fd} is n -Gorenstein if and only if Λ_{fd} is n -Gorenstein. For the comonad $\mathbf{P}_{\Lambda\text{-Mod}}$ we obtain the following generalization of (G5) from this result.

Corollary 1.1.9 (Corollary 5.2.3). *Let k be a commutative ring, and let Λ be a k -algebra which is finitely generated and projective as a k -module. Assume that*

$$\text{proj. dim } D(\Lambda)_\Lambda < \infty \quad \text{and} \quad \text{proj. dim } {}_\Lambda D(\Lambda) < \infty.$$

Then

$$\text{proj. dim } D(\Lambda)_\Lambda = \text{proj. dim } {}_\Lambda D(\Lambda).$$

Note that this dimension is n if and only if $\mathbf{P}_{\Lambda\text{-Mod}}$ is n -Gorenstein. We also obtain a version of this result for small, locally bounded, and Hom-finite categories, see Theorem 5.2.2

Finally, we get a generalization of (G6).

Theorem 1.1.10 (See Theorem 5.1.7). *Let \mathbf{P} be a generating comonad on \mathcal{A} with a Nakayama functor ν relative to \mathbf{P} . The following statements are equivalent:*

- (a) \mathbf{P} is n -Gorenstein;
- (b) $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(\mathcal{A}) = n$;
- (c) $\text{cores. dim}_{\mathcal{G}_{\mathbf{P}} \text{ inj}(\mathcal{A})}(\mathcal{A}) = n$.

The homological conjectures are among the most famous open problems in representation theory of finite-dimensional algebras. One of these conjectures is the *Gorenstein symmetry conjecture*, see [ARS]. Let Λ be a finite-dimensional algebra. The conjecture states that if ${}_\Lambda \Lambda$ has finite injective dimension, then Λ_Λ has finite injective dimension. It is known to hold if the injective dimension of ${}_\Lambda \Lambda$ is 1, see [Bo]. Motivated by our results for the comonad $\mathbf{P}_{\Lambda\text{-Mod}}$, we give a generalization of this conjecture.

Conjecture 1.1.11. *Let k be a commutative ring, and let Λ be a k -algebra which is finitely generated and projective as a k -module. Then*

$$\text{proj. dim } {}_\Lambda D(\Lambda) < \infty \quad \implies \quad \text{proj. dim } D(\Lambda)_\Lambda < \infty$$

The paper is organized as follows. In Section 2 we recall the necessary notions and results on monads and comonads that we need. In Section 3 we introduce Gorenstein flat objects for comonads and Gorenstein injective objects for monads. We also show that they form a resolving and coresolving subcategory, respectively. In Section 4 we define Nakayama functors relative to a comonad, provide examples of such comonads, and prove some basic results about them. In particular, in Subsection 4.3 we show that if the Nakayama functor exists, then it is unique and can be constructed using only the adjoints of the comonad. In Section 5 we define what it means for a comonad to be Gorenstein. In Theorem 5.1.3 we show that the Gorenstein flat objects have a simple description when the comonad is Gorenstein, and in Theorem 5.1.7 we prove an analogue of Iwanaga's result on the injective dimension of an Iwanaga-Gorenstein algebra.

1.2. Conventions. In what follows, k denotes a commutative ring, and all functors and categories are linear over k . We let $\text{proj } k$ denote the category of finitely generated projective k -modules. For a ring Λ we let $\Lambda\text{-Mod}$ and $\Lambda\text{-mod}$ denote the category of left Λ -modules and finitely presented left Λ -modules, respectively. Similarly, we let $\text{Mod-}\Lambda$ and $\text{mod-}\Lambda$ denote the category of right modules and the category of finitely presented right Λ -modules, respectively. The Hom-sets in these categories are denoted by $\text{Hom}_{\Lambda\text{-Mod}}(-, -)$ and $\text{Hom}_{\text{Mod-}\Lambda}(-, -)$. For k we simply write $\text{Hom}_k(-, -)$. We always let \mathcal{A} and \mathcal{B} denote abelian categories. The subcategory of projective objects in \mathcal{A} is denoted by $\text{Proj}(\mathcal{A})$, and the subcategory of injective objects in \mathcal{A} by $\text{Inj}(\mathcal{A})$. All subcategories are assumed to be full. By a resolving (resp. coresolving) subcategory we mean a subcategory closed under extensions, direct summands, cokernels (resp. kernels), and which contains a generating set (resp. cogenerating set), following the conventions in [S, Definition 2.1]. For a natural transformation $\eta: F \rightarrow F'$ we let $\eta_c: F(c) \rightarrow F'(c)$ denote the component of η at c . The natural transformation obtained by precomposing with a functor G is denoted by $\eta_G: F \circ G \rightarrow F' \circ G$.

2. PRELIMINARIES

2.1. Transformation of adjoints. An adjunction $(L, R, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{B}$ is given by a pair of functors $L: \mathcal{A} \rightarrow \mathcal{B}$ and $R: \mathcal{B} \rightarrow \mathcal{A}$ together with an isomorphism

$$\phi: \mathcal{A}(L(A), B) \rightarrow \mathcal{B}(A, R(B))$$

for all pairs $B \in \mathcal{B}$ and $A \in \mathcal{A}$, which is natural in A and B . We also write $L \dashv R$ if there exists such an adjunction. Here $\beta_B := \phi^{-1}(1_{R(B)}): LR(B) \rightarrow B$ and $\alpha_A := \phi(1_{L(A)}): A \rightarrow RL(A)$ are the counit and the unit of the adjunction. They satisfy the *triangular identities*, i.e $R(\beta_B) \circ \alpha_{R(B)} = 1_{R(B)}$ and $\beta_{L(A)} \circ L(\alpha_A) = 1_{L(A)}$. By naturality we also have that $\phi(f) = R(f) \circ \alpha_A$ and $\phi^{-1}(g) = \beta_B \circ L(g)$ for any morphisms $f: L(A) \rightarrow B$ and $g: A \rightarrow R(B)$.

Assume $(L_1, R_1, \phi_1, \alpha_1, \beta_1): \mathcal{A} \rightarrow \mathcal{B}$ and $(L_2, R_2, \phi_2, \alpha_2, \beta_2): \mathcal{A} \rightarrow \mathcal{B}$ are adjunctions. Following [Ma], we say that two natural transformations $\sigma: L_1 \rightarrow L_2$ and $\tau: R_2 \rightarrow R_1$ are *conjugate* (for the given adjunctions) if the square

$$\begin{array}{ccc} \mathcal{A}(L_2(A), B) & \xrightarrow{\phi_2} & \mathcal{B}(A, R_2(B)) \\ \downarrow - \circ \sigma_A & & \downarrow \tau_B \circ - \\ \mathcal{A}(L_1(A), B) & \xrightarrow{\phi_1} & \mathcal{B}(A, R_1(B)) \end{array} \quad (2.1.1)$$

commutes for all pairs $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proposition 2.1.2. *Let $(L_i, R_i, \phi_i, \alpha_i, \beta_i): \mathcal{A} \rightarrow \mathcal{B}$ be adjunctions for $1 \leq i \leq 3$. The following hold:*

- (i) If $\sigma: L_1 \rightarrow L_2$ is a natural transformation, then there exists a unique natural transformation $\tau: R_2 \rightarrow R_1$ which is conjugate to σ ;
- (ii) If $\tau: R_2 \rightarrow R_1$ is a natural transformation, then there exists a unique natural transformation $\sigma: L_1 \rightarrow L_2$ which is conjugate to τ ;
- (iii) If $\sigma_1: L_1 \rightarrow L_2$ is conjugate to $\tau_1: R_2 \rightarrow R_1$ and $\sigma_2: L_2 \rightarrow L_3$ is conjugate to $\tau_2: R_3 \rightarrow R_2$, then $\sigma_2 \circ \sigma_1: L_1 \rightarrow L_3$ is conjugate to $\tau_1 \circ \tau_2: R_3 \rightarrow R_1$.

Proof. This follows from [Ma, Theorem IV.7.2] \square

Explicitly, if $\sigma: L_1 \rightarrow L_2$ is a natural transformation, then by putting $A = R_2(B)$ in 2.1.1 we see that the conjugate $\tau: R_2 \rightarrow R_1$ is given by $\tau_B = \phi_1(\phi_2^{-1}(1_{R_2(B)}) \circ \sigma_{R_2(B)})$. Simplifying this gives

$$\tau_B = R_1((\beta_2)_B) \circ R_1(\sigma_{R_2(B)}) \circ (\alpha_1)_{R_2(B)}: R_2(B) \rightarrow R_1(B). \quad (2.1.3)$$

Conversely, if $\tau: R_2 \rightarrow R_1$ is a natural transformation, then the conjugate $\sigma: L_1 \rightarrow L_2$ is given by

$$\sigma_A = (\beta_1)_{L_2(A)} \circ L_1(\tau_{L_2(A)}) \circ L_1((\alpha_2)_A): L_1(A) \rightarrow L_2(A). \quad (2.1.4)$$

2.2. Monads and comonads.

Definition 2.2.1.

- (i) A *monad* on \mathcal{A} is a tuple $\mathbb{T} = (T, \eta, \mu)$, where $T: \mathcal{A} \rightarrow \mathcal{A}$ is a functor and $\eta: 1_{\mathcal{A}} \rightarrow T$ and $\mu: T \circ T \rightarrow T$ are natural transformations such that the diagrams

$$\begin{array}{ccc} TTT & \xrightarrow{\mathbb{T}(\mu)} & TT \\ \downarrow \mu_T & & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array} \quad \begin{array}{ccccc} & & T & & \\ & 1_T \nearrow & \uparrow \mu & \nwarrow 1_T & \\ T & \xrightarrow{T(\eta)} & TT & \xleftarrow{\eta_T} & T \end{array}$$

commute.

- (ii) A *comonad* on \mathcal{A} is a tuple $\mathbb{S} = (S, \epsilon, \Delta)$, where $S: \mathcal{A} \rightarrow \mathcal{A}$ is a functor and $\epsilon: S \rightarrow 1_{\mathcal{A}}$ and $\Delta: S \rightarrow S \circ S$ are natural transformations such that the diagrams

$$\begin{array}{ccc} S & \xrightarrow{\Delta} & SS \\ \downarrow \Delta & & \downarrow \Delta_S \\ SS & \xrightarrow{S(\Delta)} & SSS \end{array} \quad \begin{array}{ccccc} & & S & & \\ & 1_S \nearrow & \downarrow \Delta & \nwarrow 1_S & \\ S & \xleftarrow{S(\epsilon)} & SS & \xrightarrow{\epsilon_S} & S \end{array}$$

commute.

Note that a *comonad* $\mathbb{T} = (T, \eta, \mu)$ on \mathcal{A} is the same as a monad on \mathcal{A}^{op} .

Remark 2.2.2. The category of endofunctors on \mathcal{A} is a monoidal category. The product is given by composition, and the unit object is the identity functor. A monad is just a monoid and a comonad is just a comonoid in this monoidal category.

Example 2.2.3. Assume $(L, R, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{B}$ is an adjunction. Then the tuple $(R \circ L, \alpha, R(\beta_L))$ is a monad on \mathcal{A} , and the tuple $(L \circ R, \beta, L(\alpha_R))$ is a comonad on \mathcal{B} .

We need the following category later.

Definition 2.2.4. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on \mathcal{A} . The *Kleisli category* of \mathbb{T} , denoted $\text{Kl } \mathbb{T}$, is the category consisting of the same objects as \mathcal{A} , and with morphisms $\text{Kl } \mathbb{T}(A_1, A_2) = \mathcal{A}(A_1, T(A_2))$. The composition map

$$\text{Kl } \mathbb{T}(A_2, A_3) \times \text{Kl } \mathbb{T}(A_1, A_2) \rightarrow \text{Kl } \mathbb{T}(A_1, A_3)$$

sends (f, g) to the composite

$$A_1 \xrightarrow{g} T(A_2) \xrightarrow{T(f)} TT(A_3) \xrightarrow{\mu_{A_3}} T(A_3)$$

The unit morphism in $\text{Kl } \mathbb{T}(A, A)$ is given by $\eta_A: A \rightarrow T(A)$.

For a functor $L: \mathcal{B} \rightarrow \mathcal{A}$, let $\overline{\text{im}}L$ be the *full image* of L . It has the same objects as \mathcal{A} , and a morphism in $\overline{\text{im}}L$ between object X and Y is given by a morphism $L(X) \rightarrow L(Y)$ in \mathcal{B} .

Lemma 2.2.5. Let $(L, R, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{B}$ is an adjunction. Then there is an equivalence

$$\overline{\text{im}}L \cong \text{Kl}(R \circ L, \alpha, R(\beta_L))$$

acting as identity on objects, and sending a morphism $f: L(X) \rightarrow L(Y)$ to $\phi(f): X \rightarrow RL(Y)$

Proof. This follows from Proposition 4.2.1 in [Bor]. \square

Let $(F, G, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ be an adjunction. It follows that there exists an adjunction $(F^n, G^n, \phi^n, \alpha^n, \beta^n): \mathcal{A} \rightarrow \mathcal{A}$. The bijection, the unit, and the counit are given by

$$\begin{aligned} \phi^n: \mathcal{A}(F^n, -) &\xrightarrow{\phi} \mathcal{A}(F^{n-1}, G) \xrightarrow{\phi^{n-1}} \mathcal{A}(-, G^n), \\ \alpha^n &:= G^{n-1}(\alpha_{F^{n-1}}) \circ \alpha^{n-1}, \\ \beta^n &:= \beta \circ F(\beta_G^{n-1}). \end{aligned}$$

Hence, if $\mathbb{S} = (S, \epsilon, \Delta)$ is a comonad and T is an adjoint of S , then by Proposition 2.1.2 there exist unique natural transformations $TT \rightarrow T$ and $1_{\mathcal{A}} \rightarrow T$ which are conjugate to $\Delta: S \rightarrow SS$ and $\epsilon: S \rightarrow 1_{\mathcal{A}}$. Now assume that $\mathbb{S} = (S, \epsilon, \Delta)$ is a comonad on \mathcal{A} and $\mathbb{T} = (T, \eta, \mu)$ is a monad on \mathcal{A} . We say that $(\mathbb{T}, \mathbb{S}, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction if $(T, S, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction such that $\mu: TT \rightarrow T$ is conjugate to $\Delta: S \rightarrow SS$ and $\eta: 1_{\mathcal{A}} \rightarrow T$ is conjugate to $\epsilon: S \rightarrow 1_{\mathcal{A}}$. We define the adjunction $(\mathbb{S}, \mathbb{T}, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ similarly.

Proposition 2.2.6. *Let $(T_1, S_1, \phi_1, \alpha_1, \beta_1): \mathcal{A} \rightarrow \mathcal{A}$ and $(S_2, T_2, \phi_2, \alpha_2, \beta_2): \mathcal{A} \rightarrow \mathcal{A}$ be adjunctions. The following hold:*

- (i) *If $S_1 = (S_1, \epsilon_1, \Delta_1)$ is a comonad, then there exists a unique monad $T_1 = (T_1, \eta_1, \mu_1)$ such that $(T_1, S_1, \phi_1, \alpha_1, \beta_1): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction;*
- (ii) *If $T_1 = (T_1, \eta_1, \mu_1)$ is a monad, then there exists a unique comonad $S_1 = (S_1, \epsilon_1, \Delta_1)$ such that $(T_1, S_1, \phi_1, \alpha_1, \beta_1): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction;*
- (iii) *If $S_2 = (S_2, \epsilon_2, \Delta_2)$ is a comonad, then there exists a unique monad $T_2 = (T_2, \eta_2, \mu_2)$ such that $(S_2, T_2, \phi_2, \alpha_2, \beta_2): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction;*
- (iv) *If $T_2 = (T_2, \eta_2, \mu_2)$ is a monad, then there exists a unique comonad $S_2 = (S_2, \epsilon_2, \Delta_2)$ such that $(S_2, T_2, \phi_2, \alpha_2, \beta_2): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction.*

Proof. Parts (i) and (ii) follow from [EM, Proposition 3.1], and parts (iii) and (iv) are proved similarly. \square

From (2.1.4) we get that

$$\mu_1 = (\beta_1)_{T_1} \circ T_1((\beta_1)_{S_1 T_1}) \circ T_1 T_1((\Delta_1)_{T_1}) \circ T_1 T_1(\alpha_1) \quad (2.2.7)$$

and

$$\eta_1 = (\epsilon_1)_T \circ \alpha_1. \quad (2.2.8)$$

In the following we let $\text{Ker } \epsilon: \mathcal{A} \rightarrow \mathcal{A}$ and $\text{Coker } \eta: \mathcal{A} \rightarrow \mathcal{A}$ denote the functors sending $A \in \mathcal{A}$ to $\text{Ker } \epsilon_A$ and $\text{Coker } \eta_A$ respectively.

Proposition 2.2.9. *Let $(T, S, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ be an adjunction between the monad $T = (T, \eta, \mu)$ and the comonad $S = (S, \epsilon, \Delta)$. Then the functor $\text{Coker } \eta$ is left adjoint to $\text{Ker } \epsilon$.*

Proof. Let $A_1, A_2 \in \mathcal{A}$ be arbitrary. We have exact sequences

$$\begin{aligned} A_1 &\xrightarrow{\eta_{A_1}} T(A_1) \rightarrow \text{Coker } \eta_{A_1} \rightarrow 0 \\ 0 &\rightarrow \text{Ker } \epsilon_{A_2} \rightarrow S(A_2) \xrightarrow{\epsilon_{A_2}} A_2. \end{aligned}$$

Applying $\mathcal{A}(-, A_2)$ to the first sequence and $\mathcal{A}(A_1, -)$ to the second sequence gives a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A}(\text{Coker } \eta_{A_1}, A_2) & \longrightarrow & \mathcal{A}(T(A_1), A_2) \xrightarrow{- \circ \eta_{A_1}} \mathcal{A}(A_1, A_2) \\ & & \downarrow \cong & & \downarrow \phi \\ 0 & \longrightarrow & \mathcal{A}(A_1, \text{Ker } \epsilon_{A_2}) & \longrightarrow & \mathcal{A}(A_1, S(A_2)) \xrightarrow{\epsilon_{A_2} \circ -} \mathcal{A}(A_1, A_2) \end{array}$$

with exact rows. It only remains to show that the right square commutes, since in this case we get an induced natural isomorphism $\mathcal{A}(\text{Coker } \eta_{A_1}, A_2) \cong \mathcal{A}(A_1, \text{Ker } \epsilon_{A_2})$. To this end, let $f \in \mathcal{A}(A_1, S(A_2))$, and note that $\phi^{-1}(f) = \beta_{A_2} \circ T(f)$. Furthermore, by (2.2.8) we know that $\eta_{A_1} = \epsilon_{T(A_1)} \circ \alpha_{A_1}$. Hence,

we only need to show that the compositions $\epsilon_{A_2} \circ f$ and $\beta_{A_2} \circ T(f) \circ \epsilon_{T(A_1)} \circ \alpha_{A_1}$ are the same. By naturality of ϵ we have that $T(f) \circ \epsilon_{T(A_1)} = \epsilon_{TS(A_2)} \circ ST(f)$ and $\beta_{A_2} \circ \epsilon_{TS(A_2)} = \epsilon_{A_2} \circ S(\beta_{A_2})$. Furthermore, we also have that $ST(f) \circ \alpha_{A_1} = \alpha_{S(A_2)} \circ f$ by naturality of α . Hence,

$$\begin{aligned} \beta_{A_2} \circ T(f) \circ \epsilon_{T(A_1)} \circ \alpha_{A_1} &= \beta_{A_2} \circ \epsilon_{TS(A_2)} \circ ST(f) \circ \alpha_{A_1} \\ &= \epsilon_{A_2} \circ S(\beta_{A_2}) \circ ST(f) \circ \alpha_{A_1} = \epsilon_{A_2} \circ S(\beta_{A_2}) \circ \alpha_{S(A_2)} \circ f = \epsilon_{A_2} \circ f \end{aligned}$$

where the last equality follows from the triangle identities of the adjunction. This proves the claim. \square

A *morphism of comonads* $\zeta: (S_1, \epsilon_1, \Delta_1) \rightarrow (S_2, \epsilon_2, \Delta_2)$ is given by a natural transformation $\zeta: S_1 \rightarrow S_2$ satisfying $\Delta_2 \circ \zeta = \zeta^2 \circ \Delta_1$ and $\epsilon_1 = \epsilon_2 \circ \zeta$, where $\zeta^2 = S_2(\zeta) \circ \zeta_{S_1} = \zeta_{S_2} \circ S_1(\zeta)$. Dually, a *morphism of monads* $\delta: (T_1, \eta_1, \mu_1) \rightarrow (T_2, \delta_2, \mu_2)$ is given by a natural transformation $\delta: T_1 \rightarrow T_2$ satisfying $\mu_2 \circ \delta^2 = \delta \circ \mu_1$ and $\eta_2 = \delta \circ \eta_1$, where $\delta^2 = T_2(\delta) \circ \delta_{T_1} = \delta_{T_2} \circ T_1(\delta)$.

2.3. Comonadic homology. We need the theory of comonadic homology as introduced in [BB] to give a useful definition of left derived functor when there aren't enough projectives or injectives. See 2.4.11 for an example. For an introduction to this theory see section 8.7 in [W]. Here we recall the basic definitions that we need. Our terminology differs from [BB] and [W] in that we use the terms comonad and monad rather than cotriple and triple.

Assume we have a comonad $S = (S, \epsilon, \Delta)$ on \mathcal{A} . For each $n \geq 1$ we have maps

$$\partial_i = S^i(\epsilon_{S^{n-i}}): S^{n+1} \rightarrow S^n \quad \text{for } 0 \leq i \leq n$$

where $S^i = S \circ S \circ \dots \circ S$ denotes the composition taken i times. This gives a complex

$$N(S^*) = \dots \xrightarrow{d_{n+1}} S^{n+1} \xrightarrow{d_n} S^n \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} S$$

in $\mathcal{A}^{\mathcal{A}}$, where

$$d_n = \sum_{i=0}^n (-1)^i \partial_i: S^{n+1} \rightarrow S^n$$

¹ We can augment this with $\epsilon: S \rightarrow 1_{\mathcal{A}}$ to get a complex

$$(N(S^*) \xrightarrow{\epsilon} 1_{\mathcal{A}}) = (\dots \xrightarrow{d_{n+1}} S^{n+1} \xrightarrow{d_n} S^n \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} S \xrightarrow{\epsilon} 1_{\mathcal{A}}).$$

Evaluating at an object $A \in \mathcal{A}$ gives a complex

$$N(S^*A) = \dots \xrightarrow{(d_{n+1})_A} S^{n+1}(A) \xrightarrow{(d_n)_A} S^n(A) \xrightarrow{(d_{n-1})_A} \dots \xrightarrow{(d_1)_A} S(A)$$

in \mathcal{A} , with augmentation

$$\begin{aligned} (N(S^*A) \xrightarrow{\epsilon_A} A) \\ = (\dots \xrightarrow{(d_{n+1})_A} S^{n+1}(A) \xrightarrow{(d_n)_A} S^n(A) \xrightarrow{(d_{n-1})_A} \dots \xrightarrow{(d_1)_A} S(A) \xrightarrow{\epsilon_A} A). \end{aligned}$$

¹The ∂_i are boundary maps in a simplicial object S^* in $\mathcal{A}^{\mathcal{A}}$, and via Dold-Kahn correspondence we get a complex $N(S^*)$. See [W] section 8.6 for details.

This complex is in general not exact.

Following [BB], we say an object $A \in \mathcal{A}$ is \mathbf{S} -projective if the map $\epsilon_A: S(A) \rightarrow A$ is a split epimorphism. The following lemma is well known.

Lemma 2.3.1. *An object A is \mathbf{S} -projective if and only if it is a direct summand of an object $S(A')$ for some object $A' \in \mathcal{A}$.*

Example 2.3.2. Let $(L, R, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{B}$ be an adjunction with induced comonad $\mathbf{S} = (L \circ R, \beta, L(\alpha_R))$ on \mathcal{A} . It follows from the triangle identity of the adjunction that the composition

$$R \xrightarrow{\alpha_R} R \circ L \circ R \xrightarrow{R(\beta)} R$$

is the identity. Hence, any object $R(B)$ for $B \in \mathcal{B}$ is \mathbf{S} -projective, and the \mathbf{S} -projective objects are precisely the direct summands of objects of the form $R(B)$.

Lemma 2.3.3. *Assume A is \mathbf{S} -projective. Then the complex $N(S^*A) \xrightarrow{\epsilon_A} A$ is contractible.*

Proof. This follows from the proof of Proposition 8.6.8 in [W]. □

Following [BB], we say that a sequence

$$\dots \xrightarrow{f_{-2}} A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots$$

is \mathbf{S} -exact if the sequence

$$\dots \xrightarrow{f_{-2} \circ -} \mathcal{A}(A, A_{-1}) \xrightarrow{f_{-1} \circ -} \mathcal{A}(A, A_0) \xrightarrow{f_0 \circ -} \mathcal{A}(A, A_1) \xrightarrow{f_1 \circ -} \dots$$

is exact for all \mathbf{S} -projective objects $A \in \mathcal{A}$.

Lemma 2.3.4. *The complex $N(S^*A) \xrightarrow{\epsilon_A} A$ is \mathbf{S} -exact for all $A \in \mathcal{A}$.*

Proof. See 4.2 in [BB] □

For a functor $E: \mathcal{A} \rightarrow \mathcal{B}$ we let $N(ES^*A)$ denote the complex

$$\dots \xrightarrow{E((d_{n+1})_A)} ES^{n+1}(A) \xrightarrow{E(d_n)_A} ES^n(A) \xrightarrow{E((d_{n-1})_A)} \dots \xrightarrow{E((d_1)_A)} ES(A)$$

obtained by applying E to $N(S^*A)$. Its augmentation

$$\dots \xrightarrow{E((d_n)_A)} ES^n(A) \xrightarrow{E((d_{n-1})_A)} \dots \xrightarrow{E((d_1)_A)} ES(A) \xrightarrow{E(\epsilon_A)} E(A)$$

is denoted by $N(ES^*A) \rightarrow E(A)$.

Definition 2.3.5. Let $\mathbf{S} = (S, \epsilon, \Delta)$ be a comonad on \mathcal{A} , let $E: \mathcal{A} \rightarrow \mathcal{B}$ be a functor, and let $A \in \mathcal{A}$. The *comonad homology of A with coefficients in E* are the homology groups $H_n(A; E) := H_n(N(ES^*A))$.

We have an induced morphism

$$\lambda_{A,E}: H_0(A; E) \rightarrow E(A).$$

However, this is in general not an isomorphism.

A morphism $A \xrightarrow{f} A'$ in \mathcal{A} induces a morphism $N(S^*f): N(S^*A) \rightarrow N(S^*A')$ of complexes, and hence descends to a morphism

$$H_n(A; E) \xrightarrow{H_n(f; E)} H_n(A'; E)$$

on homology. For $n = 0$ we have $\lambda_{A',E} \circ H_0(f; E) = E(f) \circ \lambda_{A,E}$.

The following results shows that the \mathbf{S} -projective objects behave a bit like projective objects in that they don't have any nonzero homology with coefficient in E .

Lemma 2.3.6. *Assume A is \mathbf{S} -projective. Then*

$$H_n(A; E) = 0 \quad \text{for } n \geq 1$$

and $\lambda_{A,E}: H_0(A; E) \rightarrow E(A)$ is an isomorphism.

Proof. See 4.3 in [BB]. □

Now assume that $\mathbf{T} = (T, \eta, \mu)$ is a monad in \mathcal{A} , and let \mathbf{T}^{op} denote the corresponding comonad in \mathcal{A}^{op} . The \mathbf{T} -*injective* objects in \mathcal{A} are defined to be the \mathbf{T}^{op} -projective objects in \mathcal{A}^{op} . Given $A \in \mathcal{A}$ and a functor $E: \mathcal{A} \rightarrow \mathcal{B}$, the *monad cohomology of A with coefficients in E* is

$$H^n(A; E) := H_n(A; E^{\text{op}})$$

where $E^{\text{op}}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$ is the induced functor on the opposite categories, and where the homology is taken with respect to \mathbf{T}^{op} . The dual statements of this section then hold.

2.4. Exact and generating comonads.

Let $\mathbf{P} = (P, \epsilon, \Delta)$ be a comonad on \mathcal{A} . We want to work with \mathbf{P} -projective objects as if they were projective objects. In particular, we want to be able to take resolutions of any object in \mathcal{A} by \mathbf{P} -projective objects. To this end, we need to require that the collection of \mathbf{P} -projective objects is generating. This means that for any object $A \in \mathcal{A}$ there exists an object $A' \in \mathcal{A}$ and an epimorphism $f: P(A') \rightarrow A$. On the other hand, since $\epsilon_A \circ P(f) \circ \Delta_{A'} = f \circ \epsilon_{A'} \circ \Delta_{A'} = f$, it follows that ϵ_A must be surjective. This motivates the following definition.

Definition 2.4.1. Let $\mathbf{P} = (P, \epsilon, \Delta)$ be a comonad on \mathcal{A} . We say that \mathbf{P} is *generating* if $\epsilon: P \rightarrow 1_{\mathcal{A}}$ is a surjection.

By the discussion above we see that \mathbf{P} is generating if and only if the collection of \mathbf{P} -projective objects is a generating set in \mathcal{A} . Furthermore, since ϵ is surjective, it follows that the functor P is faithful.

Example 2.4.2. Let $(L, R, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{B}$ be an adjunction, and let $P := (L \circ R, \beta, L(\alpha_R))$ be the corresponding comonad on \mathcal{A} . It is well known that the counit of the adjunction is surjective if and only if the right adjoint is a faithful functor [Ma, Theorem IV.3.1]. Hence, P is generating if and only if R is faithful.

The following lemma follows immediately from surjectivity of $\epsilon: P \rightarrow 1_{\mathcal{A}}$.

Lemma 2.4.3. *Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} . Then any projective object in \mathcal{A} is P -projective.*

Lemma 2.4.4. *Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} . Then the complex $N(P_*A) \xrightarrow{\epsilon_A} A$ is acyclic for all $A \in \mathcal{A}$.*

Proof. We know by Lemma 2.3.4 that the complex $\mathcal{A}(A', N(P_*A)) \xrightarrow{\epsilon_{A'} \circ -} \mathcal{A}(A', A)$ is exact for all P -projective objects $A' \in \mathcal{A}$. Since the P -projective objects form a generating set in \mathcal{A} , the result immediately follows. \square

Lemma 2.4.5. *Let $P = (P, \epsilon, \Delta)$ be a generating comonad on \mathcal{A} , and let $E: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor. Then*

$$\lambda_{A,E}: H_0(A; E) \rightarrow E(A)$$

is an isomorphism.

Proof. This follows immediately from Lemma 2.4.4. \square

We want a short exact sequence

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

to induce a long exact sequence in homology. To this end, we need to require that the sequence

$$0 \rightarrow P(A_1) \xrightarrow{P(f)} P(A_2) \xrightarrow{P(g)} P(A_3) \rightarrow 0 \quad (2.4.6)$$

on P -projective covers is exact. This motivates the following definition.

Definition 2.4.7. Let $P = (P, \epsilon, \Delta)$ be a comonad on \mathcal{A} . We say that P is *exact* if the functor $P: \mathcal{A} \rightarrow \mathcal{A}$ is exact.

Lemma 2.4.8. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be an exact sequence in \mathcal{A} , and let $P = (P, \epsilon, \Delta)$ be a generating and exact comonad on \mathcal{A} . Furthermore, let $E: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor such that the composition $E \circ P: \mathcal{A} \rightarrow \mathcal{B}$ is exact. The following hold:

(i) *We have a short exact sequence*

$$0 \rightarrow N(P_*A_1) \xrightarrow{N(P_*f)} N(P_*A_2) \xrightarrow{N(P_*g)} N(P_*A_3) \rightarrow 0$$

of complexes in \mathcal{A} ;

(ii) *There is a long exact sequence*

$$\begin{aligned} \dots &\xrightarrow{H_{i+1}(g;E)} H_{i+1}(A_3; E) \xrightarrow{\partial} H_i(A_1; E) \\ &\xrightarrow{H_i(f;E)} H_i(A_2; E) \xrightarrow{H_i(g;E)} H_i(A_3; E) \xrightarrow{\partial} \dots \\ &\dots \xrightarrow{H_1(g;E)} H_1(A_3; E) \xrightarrow{\partial} E(A_1) \xrightarrow{E(f)} E(A_2) \xrightarrow{E(g)} E(A_3) \rightarrow 0 \end{aligned}$$

in \mathcal{A} .

Proof. In degree $n \geq 0$ the sequence in (i) is

$$P^{n+1}(A_1) \xrightarrow{P^{n+1}(f)} P^{n+1}(A_2) \xrightarrow{P^{n+1}(g)} P^{n+1}(A_3)$$

which is short exact since P^{n+1} is exact. This shows (i).

In order to show (ii), we apply E to the exact sequence in (i). This gives a sequence

$$N(E(P_*A_1)) \xrightarrow{N(E(P_*f))} N(E(P_*A_2)) \xrightarrow{N(E(P_*g))} N(E(P_*A_3))$$

of complexes. In degree $n \geq 0$ this is

$$0 \rightarrow EP^{n+1}(A_1) \xrightarrow{EP^{n+1}(f)} EP^{n+1}(A_2) \xrightarrow{EP^{n+1}(g)} EP^{n+1}(A_3) \rightarrow 0$$

which is short exact since $E \circ P$ is exact. Taking the induced long exact sequence in homology and using Lemma 2.4.5 shows (ii). \square

The criteria that $E \circ P$ is exact should be interpreted as saying that E is exact on the \mathbf{P} -projective covers in (2.4.6). It is a necessary property for the existence of a long exact sequence, since $H_0(A; E) \cong E(A)$ and $H_i(A; E) = 0$ when $i > 0$ and A is \mathbf{P} -projective by Lemma 2.3.6.

An object $A \in \mathcal{A}$ is called *E-acyclic* if $H_n(A; E) = 0$ for all $n \geq 1$. An *E-acyclic resolution* of A is an exact sequence $A_\bullet \rightarrow A$ with all A_i being *E-acyclic*.

Corollary 2.4.9. *Let $\mathbf{P} = (P, \epsilon, \Delta)$ be a generating and exact comonad on \mathcal{A} , and let $E: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor such that the composition $E \circ P: \mathcal{A} \rightarrow \mathcal{B}$ is exact. Then the homology $H_n(A; E)$ can be computed using any resolution of *E-acyclic* objects, that is, $H_n(A; E) \cong H_n(E(A_\bullet))$ for any *E-acyclic* resolution $A_\bullet \rightarrow A$. In particular, $H_n(A; E)$ can be computed using any resolution of A by \mathbf{P} -projective objects.*

Proof. This follows from Lemma 2.4.8 part (ii) using dimension shifting. \square

Corollary 2.4.10. *Let $\mathbf{P} = (P, \epsilon, \Delta)$ be a generating and exact comonad on \mathcal{A} , and let $E: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor such that the composition $E \circ P: \mathcal{A} \rightarrow \mathcal{B}$ is exact. Assume \mathcal{A} has enough projectives. Then*

$$H_n(A; E) \cong L_n(E)(A)$$

where $L_n(E)$ is the n th left derived functor of E .

Proof. This follows from Corollary 2.4.9 and Lemma 2.4.3. \square

If \mathcal{A} does not have enough projectives, then Corollary 2.4.10 indicates that $H_n(-; E)$ should be thought of as a replacement of the n th left derived functors of E .

Example 2.4.11. Let Λ_0 and Λ_1 be rings, and assume there is a morphism of rings $\Lambda_0 \xrightarrow{f} \Lambda_1$ which makes Λ_1 into a flat right Λ_0 -module. The restriction functor

$$f^*: \Lambda_1\text{-Mod} \rightarrow \Lambda_0\text{-Mod} \quad f^*(M) = {}_{\Lambda_0}M$$

has a left adjoint

$$f_! = \Lambda_1 \otimes_{\Lambda_0} -: \Lambda_0\text{-Mod} \rightarrow \Lambda_1\text{-Mod}$$

and the composition

$$P := f_! \circ f^*: \Lambda_1\text{-Mod} \rightarrow \Lambda_1\text{-Mod}$$

gives rise to an exact and generating comonad $\mathbf{P} = (P, \epsilon, \Delta)$, see Example 2.2.3. A module $N \in \text{Mod-}\Lambda_1$ induces a functor

$$N \otimes_{\Lambda_1} -: \Lambda_1\text{-Mod} \rightarrow \text{Mod-}k$$

and taking homology with respect to \mathbf{P} gives

$$H_n(M; N \otimes_{\Lambda_1} -) = \text{Tor}_n^{\Lambda_1/\Lambda_0}(N, M)$$

where $\text{Tor}_n^{\Lambda_1/\Lambda_0}(N, M)$ is the n th *relative Tor group*, see 8.7.5 in [W]. The composition $(N \otimes_{\Lambda_1} -) \circ P$ is exact if and only if N is flat over Λ_0 , and in this case we get

$$H_n(M; N \otimes_{\Lambda_1} -) = \text{Tor}_n^{\Lambda_1}(N, M)$$

by Corollary 2.4.10.

Now let $\mathbf{l} = (I, \eta, \mu)$ be a monad on \mathcal{A} . We say that \mathbf{l} is *cogenerating* if $\eta_A: A \rightarrow I(A)$ is a monomorphism for all $A \in \mathcal{A}$, and we say that \mathbf{l} is exact if the functor $I: \mathcal{A} \rightarrow \mathcal{A}$ is exact. The dual of the results above then hold. If \mathbf{l} is exact and cogenerating and $E: \mathcal{A} \rightarrow \mathcal{B}$ is a right exact functor such that $E \circ I$ is exact, then $H^0(A; E) \cong E(A)$, and any short exact sequence $0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$ in \mathcal{A} gives rise to a long exact sequence

$$\begin{aligned} 0 \rightarrow E(A_1) &\xrightarrow{E(f)} E(A_2) \xrightarrow{E(g)} E(A_3) \xrightarrow{\partial} H^1(A_1, E) \xrightarrow{H^1(f; E)} \dots \\ &\dots \xrightarrow{\partial} H^i(A_1, E) \xrightarrow{H^i(f; E)} H^i(A_2, E) \\ &\xrightarrow{H^i(g; E)} H^i(A_3, E) \xrightarrow{\partial} H^{i+1}(A_1, E) \xrightarrow{H^{i+1}(f; E)} \dots \end{aligned}$$

In particular, the cohomology of an object $A \in \mathcal{A}$ can be computed using any coresolution of E -acyclic objects, and in particular using a coresolution of \mathbf{l} -injective objects. If \mathcal{A} has enough injectives, then $H^n(A; E) \cong R^n(E)(A)$ where $R^n(E)$ is the n th right derived functor of E .

We need the following result later.

Lemma 2.4.12. *Let $P = (P, \epsilon, \Delta)$ be a comonad and $I = (I, \eta, \mu)$ a monad on \mathcal{A} . Assume we have an adjunction $(P, I, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$. Then P is generating if and only if I is cogenerating.*

Proof. Assume P is generating. Let $A \in \mathcal{A}$ and let $i: \text{Ker } \eta_A \xrightarrow{i} A$ be the inclusion. The composition $P(\text{Ker } \eta_A) \xrightarrow{\epsilon_{\text{Ker } \eta_A}} \text{Ker } \eta_A \xrightarrow{i} A$ corresponds to $\text{Ker } \eta_A \xrightarrow{i} A \xrightarrow{\eta_A} I(A)$ under ϕ since ϵ is conjugate to η . It follows that $i \circ \epsilon_{\text{Ker } \eta_A} = 0$. Since ϵ is an epimorphism and i is a monomorphism we get that $\text{Ker } \eta_A = 0$. Since A was arbitrary, the monad I is cogenerating. The converse is proved dually. \square

3. GORENSTEIN OBJECTS FOR COMONADS

In this section we introduce Gorenstein flat objects for comonads and Gorenstein injective objects for monads. Our main result is that these objects form a resolving and a coresolving subcategory respectively.

3.1. Gorenstein objects for comonads. Let Λ be a finite-dimensional algebra over a field k , and let

$$P_\bullet = \cdots \rightarrow P_{-1} \rightarrow P_0 \rightarrow P_1 \rightarrow \cdots$$

be long exact sequence of projective modules in $\Lambda\text{-mod}$. From [C1, Lemma 2.2.9] we know that P_\bullet is totally acyclic if and only if $D(\Lambda) \otimes_\Lambda P_\bullet$ is exact, where $D(\Lambda) = \text{Hom}_k(\Lambda, k)$ is the k -dual of Λ . Applying $\Lambda \otimes_k -$ to this, we see that P_\bullet is totally acyclic if and only if

$$\begin{aligned} & (\Lambda \otimes_k D(\Lambda)) \otimes_\Lambda P_\bullet \\ &= \cdots \rightarrow (\Lambda \otimes_k D(\Lambda)) \otimes_\Lambda P_{-1} \rightarrow (\Lambda \otimes_k D(\Lambda)) \otimes_\Lambda P_0 \rightarrow (\Lambda \otimes_k D(\Lambda)) \otimes_\Lambda P_1 \rightarrow \cdots \end{aligned}$$

is exact. On the other hand, the functor

$$(\Lambda \otimes_k D(\Lambda)) \otimes_\Lambda -: \Lambda\text{-mod} \rightarrow \Lambda\text{-mod}$$

is left adjoint to the functor

$$\text{Hom}_{\Lambda\text{-mod}}(\Lambda \otimes_k D(\Lambda), -): \Lambda\text{-mod} \rightarrow \Lambda\text{-mod}.$$

By the isomorphism

$$\text{Hom}_{\Lambda\text{-mod}}(\Lambda \otimes_k D(\Lambda), -) \cong (\Lambda \otimes_k \Lambda) \otimes_\Lambda - \cong \Lambda \otimes_k -$$

we see that this is just the comonad given in Example 2.4.11 with $\Lambda_0 = k$ and $\Lambda_1 = \Lambda$.

In general, we therefore look at exact generating comonads $P = (P, \epsilon, \Delta)$ on \mathcal{A} such that there exists an adjunction $(T, P, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$. We would like to be able to apply Lemma 2.4.8 to T , so we therefore assume that $T \circ P$ is exact. This motivates the following definition.

Definition 3.1.1. Let $P = (P, \epsilon, \Delta)$ be a generating and exact comonad on \mathcal{A} . We say that P *accommodates Gorenstein objects* if P has a left adjoint T such that $T \circ P: \mathcal{A} \rightarrow \mathcal{A}$ is exact.

The left adjoint T of P is uniquely determined up to unique isomorphism. Also, from Proposition 2.2.6 we know that there exists a unique monad $\mathbb{T} = (T, \eta, \mu)$ such that $(\mathbb{T}, P, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction.

We call an exact sequence

$$\cdots \xrightarrow{f_{-2}} A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots$$

T -exact if the sequence

$$\cdots \xrightarrow{T(f_{-2})} T(A_{-1}) \xrightarrow{T(f_{-1})} T(A_0) \xrightarrow{T(f_0)} T(A_1) \xrightarrow{T(f_1)} \cdots$$

is still exact.

Definition 3.1.2. Assume P accommodates Gorenstein objects. An object $X \in \mathcal{A}$ is *Gorenstein P -flat* if there exists a T -exact sequence

$$A_\bullet = \cdots \xrightarrow{f_{-2}} A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots$$

with $A_i \in \mathcal{A}$ being P -projective for all $i \in \mathbb{Z}$, and with $Z^0(A_\bullet) = X$. The subcategory consisting of all Gorenstein P -flat objects is denoted by $\mathcal{G}_P \text{ flat}(\mathcal{A})$.

Since T is a left adjoint, it preserves all colimits that exist in \mathcal{A} .

Assume X is Gorenstein P -flat. By Lemma 2.3.1 any P -projective object is a summand of an object of the form $P(A)$ with $A \in \mathcal{A}$. Hence, we can also find a T -exact sequence

$$\cdots \xrightarrow{f_{-2}} P(B_{-1}) \xrightarrow{f_{-1}} P(B_0) \xrightarrow{f_0} P(B_1) \xrightarrow{f_1} \cdots$$

with $Z^0(\cdots \xrightarrow{f_{-2}} P(B_{-1}) \xrightarrow{f_{-1}} P(B_0) \xrightarrow{f_0} P(B_1) \xrightarrow{f_1} \cdots) = X$.

Remark 3.1.3. Assume $\mathcal{A} = \Lambda\text{-Mod}$ is a module category for some ring Λ . Since $T: \Lambda\text{-Mod} \rightarrow \Lambda\text{-Mod}$ is colimit preserving, the Eilenberg-Watts theorem tells us that $T \cong M \otimes_\Lambda -$ for some Λ -bimodule M . The definition of Gorenstein P -flat objects then becomes reminiscent of the definition of Gorenstein flat modules (see [H, Definition 3.1]), hence the name Gorenstein P -flat.

Example 3.1.4. Let Λ be a finite-dimensional algebra over a field k , and let P be the comonad which accommodates Gorenstein objects given in the beginning of this subsection. The P -projective objects are in this case precisely the projective Λ -modules. Hence, as shown above the T -exact sequences with P -projective objects are precisely the totally acyclic complexes in $\Lambda\text{-mod}$. Therefore, the Gorenstein P -flat objects are the Gorenstein projective modules (which also coincide with the Gorenstein flat modules in this case).

Example 3.1.5. Let $f: \Lambda_0 \rightarrow \Lambda_1$ be a morphism of rings such that $(\Lambda_1)_{\Lambda_0}$ is finitely generated projective and $\text{Hom}_{\text{Mod-}\Lambda_0}(\Lambda_1, \Lambda_0)_{\Lambda_0}$ is flat. Let $P = (P, \epsilon, \Delta)$ be the generating and exact comonad on $\Lambda_1\text{-Mod}$ in Example

2.4.11, where $P(M) = f_! \circ f^*(M) = \Lambda_1 \otimes_{\Lambda_0} M$. For $N_0 \in \text{Mod-}\Lambda_0$ and $N_1 \in \Lambda_0\text{-Mod}$ we have a natural morphism

$$N_0 \otimes_{\Lambda_0} N_1 \xrightarrow{g} \text{Hom}_{\Lambda_0\text{-Mod}}(\text{Hom}_{\text{Mod-}\Lambda_0}(N_0, \Lambda_0), N_1)$$

given by $g(n_0 \otimes n_1)(h) = h(n_0) \cdot n_1$. This is an isomorphism if N_0 is a finitely generated projective Λ_0 -module. Hence, we get a natural isomorphism

$$f_! = \Lambda_1 \otimes_{\Lambda_0} - \cong \text{Hom}_{\Lambda_0\text{-Mod}}(\text{Hom}_{\text{Mod-}\Lambda_0}(\Lambda_1, \Lambda_0), -) : \Lambda_0\text{-Mod} \rightarrow \Lambda_1\text{-Mod}.$$

This implies that

$$\text{Hom}_{\text{Mod-}\Lambda_0}(\Lambda_1, \Lambda_0) \otimes_{\Lambda_1} - : \Lambda_1\text{-Mod} \rightarrow \Lambda_0\text{-Mod}$$

is left adjoint to $f_!$. The functor

$$T := \Lambda_1 \otimes_{\Lambda_0} \text{Hom}_{\text{Mod-}\Lambda_0}(\Lambda_1, \Lambda_0) \otimes_{\Lambda_1} - : \Lambda_1\text{-Mod} \rightarrow \Lambda_1\text{-Mod}$$

is therefore left adjoint to P . Since $\text{Hom}_{\text{Mod-}\Lambda_0}(\Lambda_1, \Lambda_0)_{\Lambda_0}$ is flat, the composition

$$T \circ P = \Lambda_1 \otimes_{\Lambda_0} \text{Hom}_{\text{Mod-}\Lambda_0}(\Lambda_1, \Lambda_0) \otimes_{\Lambda_0} - : \Lambda_1\text{-Mod} \rightarrow \Lambda_1\text{-Mod}$$

is exact. This shows that P accomodates Gorenstein objects.

We also have the following dual notions.

Definition 3.1.6. Let $\mathbf{l} = (I, \eta, \mu)$ be a cogenerating and exact monad on \mathcal{A} . We say that \mathbf{l} *accommodates Gorenstein objects* if I has a right adjoint S such that the composition $S \circ I$ is exact.

The right adjoint S is then unique up to unique isomorphism, and from Proposition 2.2.6 there exists a unique comonad $\mathbf{S} = (S, \epsilon, \Delta)$ such that $(\mathbf{l}, \mathbf{S}, \phi, \alpha, \beta) : \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction. Note that the monad \mathbf{l} accommodates Gorenstein objects on \mathcal{A} if and only if it accommodates Gorenstein objects as a comonad on \mathcal{A}^{op} .

Definition 3.1.7. Let $\mathbf{l} = (I, \eta, \mu)$ be a monad on \mathcal{A} which accommodates Gorenstein objects, and let S be the right adjoint to I . An object $X \in \mathcal{A}$ is *Gorenstein \mathbf{l} -injective* if there exists an S -exact sequence

$$A_{\bullet} = \cdots \xrightarrow{f_{-2}} A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots$$

in \mathcal{A} with A_i being \mathbf{l} -injective for all $i \in \mathbb{Z}$, and with $Z^0(A_{\bullet}) = X$. The category of Gorenstein \mathbf{l} -injective objects is denoted by $\mathcal{G}_1 \text{inj}(\mathcal{A})$.

Remark 3.1.8. Let Λ be a ring, and let $\mathbf{l} = (I, \eta, \mu)$ be a monad on $\Lambda\text{-Mod}$ which accommodates Gorenstein objects. Let S be the right adjoint to I . By Eilenberg-Watts theorem we have that $S \cong \text{Hom}_{\Lambda\text{-Mod}}(M, -)$ for some Λ -bimodule M since S preserves limits. Interpreting \mathbf{l} -injective objects as injective objects, we see that the definition of Gorenstein \mathbf{l} -injective objects becomes reminiscent of the definition of Gorenstein injective modules, hence the name Gorenstein \mathbf{l} -injective.

3.2. Basic properties. Our goal in this section is to show that $\mathcal{G}_P \text{ flat}(\mathcal{A})$ is a resolving subcategory of \mathcal{A} containing all P -projective objects when P is a comonad which accommodates Gorenstein objects. In order to do this we need some preparation.

Lemma 3.2.1. *Let (S, ϵ, Δ) be a comonad on \mathcal{A} , let $T = (T, \eta, \mu)$ be a monad on \mathcal{A} , and assume we have an adjunction $(T, S, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$. The following hold:*

- (i) $\epsilon_T: ST \rightarrow T$ is a split epimorphism;
- (ii) $\eta_S: S \rightarrow TS$ is a split monomorphism.

Hence, an object $A \in \mathcal{A}$ is S -projective if and only if it is T -injective.

Proof. Consider the composition

$$T \xrightarrow{T(\alpha)} TST \xrightarrow{T(\Delta_T)} TSST \xrightarrow{\beta_{ST}} ST \xrightarrow{\epsilon_T} T.$$

By naturality, we have that $\epsilon_T \circ \beta_{ST} = \beta_T \circ TS(\epsilon_T)$. Also, we have that $TS(\epsilon_T) \circ T(\Delta_T) = 1$ from the definition of a comonad. Hence

$$\epsilon_T \circ \beta_{ST} \circ T(\Delta_T) \circ T(\alpha) = \beta_T \circ T(\alpha) = 1$$

where the last equality follows from the triangle identity of the adjunction. This shows that ϵ_T is a split epimorphism. Statement (ii) is proved dually. \square

We make the following assumption for the reminder of the section.

Setting 3.2.2. We assume $P = (P, \epsilon, \Delta)$ is a comonad on \mathcal{A} which accommodates Gorenstein objects. Let T be the left adjoint of P and let $T = (T, \eta, \mu)$ be the induced monad on \mathcal{A} such that $(T, P, \phi, \alpha, \beta): \mathcal{A} \rightarrow \mathcal{A}$ is an adjunction.

We let $\Omega_P^n(\mathcal{A})$ denote the full subcategory of \mathcal{A} consisting objects $A \in \mathcal{A}$ such that there exists an T -exact sequence $0 \rightarrow A \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_n$ with Q_i being P -projective for $1 \leq i \leq n$. We set $\Omega_P^0(\mathcal{A}) = \mathcal{A}$. If Q is P -projective, then $Q \in \Omega_P^n(\mathcal{A})$ for all $n \geq 0$ by Lemma 3.2.1.

Example 3.2.3. Let P be the comonad in Example 3.1.4. Then, a module in $\Omega_P^n(\Lambda - \text{mod})$ is just a n -torsion free module as defined in [AB].

Lemma 3.2.4. *Let $g: A_1 \rightarrow Q$ be a monomorphism in \mathcal{A} where Q is a P -projective object. Furthermore, let $f: A_1 \rightarrow A_2$ be a morphism in \mathcal{A} such that $T(f): T(A_1) \rightarrow T(A_2)$ is a monomorphism. Then f is a monomorphism.*

Proof. By Lemma 2.3.1 we can assume $Q = P(A)$. Consider the inclusion $i: \text{Ker } f \rightarrow A_1$. By naturality of ϕ^{-1} we have that $\phi^{-1}(g \circ i) = \phi^{-1}(g) \circ T(i)$. Since $T(f)$ is a monomorphism and $T(f) \circ T(i) = 0$, it follows that $T(i) = 0$. Hence, the composite $\phi^{-1}(g) \circ T(i)$ is 0, and therefore $g \circ i$ is also 0. Since $g \circ i$ is a monomorphism, we get that $\text{Ker } f = 0$. \square

Lemma 3.2.5. *Let $A \in \mathcal{A}$. The following statements are equivalent:*

- (i) There exists a monomorphism $0 \rightarrow A \rightarrow Q$ where Q is \mathbf{P} -projective;
- (ii) $\eta_A: A \rightarrow T(A)$ is a monomorphism;
- (iii) $A \in \Omega_{\mathbf{P}}^1(\mathcal{A})$.

Proof. The implication (iii) \implies (i) is obvious. By the definition of a monad the map $T(\eta_A): T(A) \rightarrow TT(A)$ is a split monomorphism. Lemma 3.2.4 therefore shows that (i) \implies (ii). Finally, the implication (ii) \implies (iii) follows from Lemma 3.2.1. \square

Lemma 3.2.6. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be a T -exact sequence in \mathcal{A} with $A_1, A_3 \in \Omega_{\mathbf{P}}^1(\mathcal{A})$. Then $A_2 \in \Omega_{\mathbf{P}}^1(\mathcal{A})$.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow \eta_{A_1} & & \downarrow \eta_{A_2} & & \downarrow \eta_{A_3} & & \\ 0 & \longrightarrow & T(A_1) & \xrightarrow{T(f)} & T(A_2) & \xrightarrow{T(g)} & T(A_3) & \longrightarrow & 0 \end{array}$$

with exact rows. Since η_{A_1} and η_{A_3} are monomorphism, it follows that η_{A_2} is a monomorphism. Hence, the result follows. \square

Lemma 3.2.7. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be a T -exact sequence in \mathcal{A} with A_1 \mathbf{P} -projective and $A_2 \in \Omega_{\mathbf{P}}^1(\mathcal{A})$. Then $A_3 \in \Omega_{\mathbf{P}}^1(\mathcal{A})$.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow \eta_{A_1} & & \downarrow \eta_{A_2} & & \downarrow \eta_{A_3} & & \\ 0 & \longrightarrow & T(A_1) & \xrightarrow{T(f)} & T(A_2) & \xrightarrow{T(g)} & T(A_3) & \longrightarrow & 0 \end{array}$$

where the two rows are short exact and η_{A_1} and η_{A_2} are monomorphisms. Hence, by the snake lemma, there is a long exact sequence

$$0 \rightarrow \text{Ker } \eta_{A_3} \xrightarrow{i} \text{Coker } \eta_{A_1} \xrightarrow{h} \text{Coker } \eta_{A_2} \rightarrow \text{Coker } \eta_{A_3} \rightarrow 0.$$

Furthermore, applying T gives a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(A_1) & \xrightarrow{T(f)} & T(A_2) & \xrightarrow{T(g)} & T(A_3) & \longrightarrow & 0 \\ & & \downarrow T(\eta_{A_1}) & & \downarrow T(\eta_{A_2}) & & \downarrow T(\eta_{A_3}) & & \\ 0 & \longrightarrow & TT(A_1) & \xrightarrow{TT(f)} & TT(A_2) & \xrightarrow{TT(g)} & TT(A_3) & \longrightarrow & 0 \end{array}$$

The bottom row is exact by Lemma 2.4.8 and the fact that $H_1(Q; T) = 0$ for any \mathbf{P} -projective object Q . Also, it follows from the definition of a monad that the maps $T(\eta_{A_1})$, $T(\eta_{A_2})$ and $T(\eta_{A_3})$ are split monomorphism. Since T is right exact, we get that $\text{Coker } T(\eta_{A_i}) \cong T(\text{Coker } \eta_{A_i})$ for $1 \leq i \leq 3$. Hence, by the snake lemma, the sequence

$$0 \rightarrow T(\text{Coker } \eta_{A_1}) \xrightarrow{T(h)} T(\text{Coker } \eta_{A_2}) \rightarrow T(\text{Coker } \eta_{A_3}) \rightarrow 0$$

is exact. Since A_1 is \mathbf{P} -projective, the short exact sequence

$$0 \rightarrow A_1 \xrightarrow{\eta_{A_1}} T(A_1) \rightarrow \text{Coker } \eta_{A_1} \rightarrow 0$$

splits by Lemma 3.2.1, and $\text{Coker } \eta_{A_1}$ is therefore \mathbf{P} -projective. In particular, it is contained in $\Omega_{\mathbf{P}}^1(\mathcal{A})$. Since $T(h)$ is a monomorphism, Lemma 3.2.4 implies that h is a monomorphism. This shows that $\text{Ker } \eta_{A_3} = 0$, and the claim follows. \square

Lemma 3.2.8. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be a T -exact sequence and let $i \geq 1$ be an integer. The following hold:

- (i) *If $A_1 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ and $A_2 \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$, then $A_3 \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$;*
- (ii) *If $A_1 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ and $A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$, then $A_2 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$;*
- (iii) *If A_1 is \mathbf{P} -projective and $A_2 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$, then $A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$.*

Proof. We prove the lemma by induction on i . For $i = 1$, statement (i) is obvious, and statements (ii) and (iii) are Lemma 3.2.6 and Lemma 3.2.7 respectively.

Now assume (i), (ii), and (iii) are true for $i - 1 > 0$, and let

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0 \tag{3.2.9}$$

be an exact sequence with $A_1 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ and $A_2 \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. By assumption, there exists a T -exact sequence

$$0 \rightarrow A_1 \xrightarrow{h} Q \rightarrow \text{Coker } h \rightarrow 0$$

with Q being \mathbf{P} -projective and $\text{Coker } h \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. Let E be the pushout of f along h . We then have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 \longrightarrow 0 \\ & & \downarrow h & & \downarrow & & \downarrow 1_{A_3} \\ 0 & \longrightarrow & Q & \longrightarrow & E & \longrightarrow & A_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Coker } h & \xrightarrow{1_{\text{Coker } h}} & \text{Coker } h & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

where all rows and columns are short exact sequences. Since the upper row is T -exact and T preserves pushouts, it follows that the middle row is T -exact. Hence, by the nine lemma the middle column is also T -exact. Since $\text{Coker } h \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$ and $A_2 \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$, we get by induction on (ii) that $E \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. Finally, by induction on (iii) it follows that $A_3 \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$.

Now assume $A_1 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ and $A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ in (3.2.9). By Lemma 3.2.6 we get that $A_2 \in \Omega_{\mathbf{P}}^1(\mathcal{A})$. Hence we have exact sequences

$$0 \rightarrow A_i \xrightarrow{\eta_{A_i}} T(A_i) \rightarrow \text{Coker } \eta_{A_i} \rightarrow 0$$

for $1 \leq i \leq 3$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow \eta_{A_1} & & \downarrow \eta_{A_2} & & \downarrow \eta_{A_3} & & \\ 0 & \longrightarrow & T(A_1) & \xrightarrow{T(f)} & T(A_2) & \xrightarrow{T(g)} & T(A_3) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Coker } \eta_{A_1} & \longrightarrow & \text{Coker } \eta_{A_2} & \longrightarrow & \text{Coker } \eta_{A_3} & \longrightarrow & 0 \end{array} \quad (3.2.10)$$

where all the rows and columns are short exact sequences. Note that the two upper rows and all the columns are T -exact (as short exact sequences). Hence, the lower row is T -exact by the nine lemma. By (i) we know that $\text{Coker } \eta_{A_1} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$ and $\text{Coker } \eta_{A_3} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. Therefore, $\text{Coker } \eta_{A_2} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$ by induction on (ii), and hence $A_2 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$.

We prove (iii). Assume A_1 is \mathbf{P} -projective and $A_2 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ in (3.2.9). Consider the diagram (3.2.10) above. Note that η_{A_3} is a monomorphism by Lemma 3.2.7. By the nine lemma the lower row is therefore exact. Also, as before the lower row is T -exact by the nine lemma. On the other hand, η_{A_1} is a split monomorphism by Lemma 2.4.8. Therefore, $\text{Coker } \eta_{A_1}$ is \mathbf{P} -projective. By part (i) of this lemma we therefore get that $\text{Coker } \eta_{A_2} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. It follows by induction on (iii) that $\text{Coker } \eta_{A_3} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$, and hence $A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$. \square

Lemma 3.2.11. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be a T -exact sequence, and let $i \geq 1$ be an integer. The following holds:

- (i) *If $A_2, A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$, then $A_1 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$;*
- (ii) *If $A_2 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$ and the sequence is split exact, then $A_1, A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$.*
Hence, $\Omega_{\mathbf{P}}^i(\mathcal{A})$ is closed under direct summands.

Proof. We prove this by induction. For $i = 1$ both statements are obvious, so assume $i > 1$. In both cases we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 \longrightarrow 0 \\
 & & \downarrow \eta_{A_1} & & \downarrow \eta_{A_2} & & \downarrow \eta_{A_3} \\
 0 & \longrightarrow & T(A_1) & \xrightarrow{T(f)} & T(A_2) & \xrightarrow{T(g)} & T(A_3) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Coker } \eta_{A_1} & \longrightarrow & \text{Coker } \eta_{A_2} & \longrightarrow & \text{Coker } \eta_{A_3} \longrightarrow 0
 \end{array}$$

where the rows and columns are short exact sequences. Also, in both cases the two upper rows and all the columns are T -exact as short exact sequences. Therefore, by the nine lemma the lower sequence is T -exact.

Assume $A_2, A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$. By Lemma 3.2.8 part (i) we get that $\text{Coker } \eta_{A_2}, \text{Coker } \eta_{A_3} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. Hence, by induction it follows that $\text{Coker } \eta_{A_1} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$, and therefore $A_1 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$. This shows (i).

For (ii), note that the sequence

$$0 \rightarrow \text{Coker } \eta_{A_1} \rightarrow \text{Coker } \eta_{A_2} \rightarrow \text{Coker } \eta_{A_3} \rightarrow 0$$

is split exact since the other two horizontal sequences are split exact. Also, $\text{Coker } \eta_{A_2} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$ by Lemma 3.2.8 part (i). Hence, by induction we get that $\text{Coker } \eta_{A_1}, \text{Coker } \eta_{A_3} \in \Omega_{\mathbf{P}}^{i-1}(\mathcal{A})$. This implies that $A_1, A_3 \in \Omega_{\mathbf{P}}^i(\mathcal{A})$. \square

Now let $\Omega_{\mathbf{P}}^\infty(\mathcal{A})$ be the full subcategory of \mathcal{A} consisting of object A such that there exists a T -exact sequence

$$0 \rightarrow A \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_n \rightarrow \cdots$$

where Q_i is \mathbf{P} -projective for all $i \geq 1$.

Lemma 3.2.12. *We have*

$$\Omega_{\mathbf{P}}^\infty(\mathcal{A}) = \bigcap_{n \geq 1} \Omega_{\mathbf{P}}^n(\mathcal{A}).$$

Proof. We only need to show that if $A \in \bigcap_{n \geq 1} \Omega_{\mathbf{P}}^n(\mathcal{A})$, then $A \in \Omega_{\mathbf{P}}^\infty(\mathcal{A})$. To this end, note that by Lemma 3.2.8 part (i) we have a T -exact sequence

$$0 \rightarrow A \xrightarrow{\eta_A} T(A) \rightarrow \text{Coker } \eta_A \rightarrow 0$$

where $\text{Coker } \eta_A \in \bigcap_{n \geq 1} \Omega_{\mathbf{P}}^n(\mathcal{A})$. Iterating this construction proves the claim. \square

Lemma 3.2.13. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be a T -exact sequence. The following holds:

- (i) If $A_1 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$ and $A_2 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$, then $A_3 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$;
- (ii) If $A_1 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$ and $A_3 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$, then $A_2 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$;
- (iii) If $A_2 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$ and $A_3 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$, then $A_1 \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$;
- (iv) $\Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$ is closed under direct summands.

Proof. This follows immediately from Lemma 3.2.8, Lemma 3.2.11, and Lemma 3.2.12. \square

Let $T\text{-Acy}$ denote the full subcategory of \mathcal{A} consisting of objects A such that

$$H_n(A, T) = 0 \quad \text{for all } n \geq 1.$$

It is easy to see that $T\text{-Acy}$ is closed under direct summands, extensions, and kernels of epimorphisms. Also, if $X \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$, then $X \in T\text{-Acy}$.

The following lemma is the analogue of the equivalence between (1) and (2) in Lemma 2.1.4 in [C1].

Lemma 3.2.14. *We have an equality*

$$\mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A}) = T\text{-Acy} \cap \Omega_{\mathbf{P}}^{\infty}(\mathcal{A}).$$

Proof. We only need to show that if $X \in T\text{-Acy} \cap \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$, then $X \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$. Choose a T -exact sequence

$$0 \rightarrow X \rightarrow Q_0 \rightarrow Q_1 \rightarrow \cdots$$

and a long exact sequence

$$\cdots \rightarrow Q_{-2} \rightarrow Q_{-1} \rightarrow X \rightarrow 0$$

with Q_i being \mathbf{P} -projective. Since $X \in T\text{-Acy}$, the last sequence is also T -exact. Gluing these two sequence together gives a T -exact sequence Q_{\bullet} with $Z^0(Q_{\bullet}) = X$, and hence $X \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$. \square

We can finally prove that $\mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$ is resolving.

Theorem 3.2.15. *Let*

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

be a short exact sequence. The following holds:

- (i) If $A_1 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$ and $A_3 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$, then $A_2 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$;
- (ii) If $A_2 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$ and $A_3 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$, then $A_1 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$;
- (iii) If the sequence is T -exact, $A_1 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$, and $A_2 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$, then $A_3 \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$;
- (iv) $\mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$ is closed under direct summands;

Proof. Note that in all four cases the short exact sequence is T -exact since $\mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A}) \subset T\text{-Acy}$. The statements follows then from Lemma 3.2.13 and 3.2.14 \square

Proposition 3.2.16. *If Q is \mathbf{P} -projective, then $Q \in \mathcal{G}_{\mathbf{P}}\text{flat}(\mathcal{A})$.*

Proof. Obviously $Q \in T\text{-Acy}$ and $Q \in \Omega_{\mathbf{P}}^{\infty}(\mathcal{A})$. The claim follows therefore from Lemma 3.2.14. \square

Corollary 3.2.17. *The subcategory $\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$ is resolving.*

Proof. This follows immediately from Theorem 3.2.15, Proposition 3.2.16, and the fact that the \mathbf{P} -projective object form a generating set in \mathcal{A} . \square

We also have the following dual result.

Theorem 3.2.18. *Let $\mathbf{l} = (I, \eta, \mu)$ be a monad on \mathcal{A} which accommodates Gorenstein objects, and let S be the right adjoint to I . The following holds:*

- (i) $\mathcal{G}_{\mathbf{l}} \text{ inj}(\mathcal{A})$ is a coresolving subcategory of \mathcal{A} containing all the \mathbf{l} -injective objects;
- (ii) Assume there exists an S -exact sequence

$$0 \rightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \rightarrow 0$$

with $A_2, A_3 \in \mathcal{G}_{\mathbf{l}} \text{ inj}(\mathcal{A})$. Then $A_1 \in \mathcal{G}_{\mathbf{l}} \text{ inj}(\mathcal{A})$.

Proof. This follows from Theorem 3.2.15, Proposition 3.2.16, and Corollary 3.2.17 applied to \mathcal{A}^{op} . \square

3.3. Dimension with respect to $\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$. We can define the *resolution dimension* $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(A)$ of any object $A \in \mathcal{A}$ with respect to $\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$. This is a general construction which can be made for any resolving subcategory of an abelian category, see [S]. Explicitly, it is the smallest integer $n \geq 0$ such that there exists a long exact sequence

$$0 \rightarrow X_n \rightarrow \cdots X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$$

where $X_i \in \mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$ for $0 \leq i \leq n$. We write $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(A) = \infty$ if there doesn't exist such a resolution. If $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(A) = n$, the dimension can be computed using any resolution of A by objects in $\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$, i.e for any exact sequence

$$0 \rightarrow X'_n \rightarrow \cdots X'_1 \rightarrow X'_0 \rightarrow A \rightarrow 0$$

with $X'_i \in \mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$ for all $0 \leq i \leq n-1$ we get that $X'_n \in \mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$, see [S, Proposition 2.3]. The *global resolution dimension* $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(\mathcal{A})$ of \mathcal{A} is defined as the supremum of $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(A)$ over all $A \in \mathcal{A}$.

Proposition 3.3.1. *We have $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq n$ if and only if the following holds:*

- (i) $H_i(A; T) = 0$ for all $i \geq n+1$ and all $A \in \mathcal{A}$;
- (ii) $\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A}) = T\text{-Acy}$.

Proof. Let $A \in \mathcal{A}$ be arbitrary, and let

$$0 \rightarrow X_n \rightarrow \cdots X_1 \rightarrow X_0 \rightarrow A \rightarrow 0 \tag{3.3.2}$$

be an exact sequence with $X_i \in \mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$ for all $0 \leq i \leq n-1$.

If \mathcal{A} satisfies (i), then by Lemma 2.4.8 and dimension shifting this implies $X_n \in L\text{-Acy}$. If \mathcal{A} also satisfy (ii), we get that $X_n \in \mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})$, and so $\text{res. dim}_{\mathcal{G}_{\mathbf{P}} \text{ flat}(\mathcal{A})}(A) \leq n$.

For the converse, assume $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq n$. Then $X_n \in \mathcal{G}_P \text{ flat}(\mathcal{A})$, and therefore

$$H_i(A; T) \cong H_{i-n}(X_n; T) = 0 \quad \text{for all } i \geq n+1$$

by dimension shifting and Lemma 2.4.8. This shows (i). For (ii), assume that $A \in T\text{-Acy}$. Then the sequence (3.3.2) is T -exact, and repeated use of Theorem 3.2.15 part (iii) therefore shows that $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$. \square

Proposition 3.3.3. *We have $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) = 0$ if and only if \mathbb{T} is cogenerating and exact.*

Proof. Since \mathbb{T} is cogenerating, it follows that $\Omega_P^\infty(\mathcal{A}) = \mathcal{A}$. Furthermore, since \mathbb{T} is exact, it follows that $T\text{-Acy} = \mathcal{A}$. Hence, by Lemma 3.2.14 we get that $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) = 0$. The converse is obvious. \square

Let \mathbb{I} be a monad on \mathcal{A} which accommodates Gorenstein objects. We can dually define the *coresolution dimension* $\text{cores. dim}_{\mathcal{G}_\mathbb{I} \text{ inj}(\mathcal{A})}(A)$ for an object $A \in \mathcal{A}$ and the *global coresolution dimension* $\text{cores. dim}_{\mathcal{G}_\mathbb{I} \text{ inj}(\mathcal{A})}(\mathcal{A})$ for the category \mathcal{A} . The dual of Proposition 3.3.1 and 3.3.3 gives the following.

Proposition 3.3.4. *Let $\mathbb{I} = (I, \eta, \mu)$ be a monad on \mathcal{A} which accommodates Gorenstein objects, and let S be the right adjoint to I . We have $\text{cores. dim}_{\mathcal{G}_\mathbb{I} \text{ inj}(\mathcal{A})}(\mathcal{A}) \leq n$ if and only if the following holds:*

- (i) $H^i(A; S) = 0$ for all $i \geq n+1$ and all $A \in \mathcal{A}$;
- (ii) $\mathcal{G}_\mathbb{I} \text{ inj}(\mathcal{A}) = S\text{-Acy}$.

Proposition 3.3.5. *Let \mathbb{I} be a monad on \mathcal{A} which accommodates Gorenstein objects, and let \mathbb{S} be the adjoint comonad to \mathbb{I} . Then the equality $\text{cores. dim}_{\mathcal{G}_\mathbb{I} \text{ inj}(\mathcal{A})}(\mathcal{A}) = 0$ holds if and only if \mathbb{S} is generating and exact.*

4. NAKAYAMA FUNCTOR FOR COMONADS

4.1. Definition and examples. Often a category \mathcal{A} is not only equipped with a comonad \mathbb{P} which accommodates Gorenstein objects, but also with a functor $\nu: \mathcal{A} \rightarrow \mathcal{A}$ which behaves like a Nakayama functor. This is formalised in the following definition.

Definition 4.1.1. Let \mathcal{A} be an abelian category with a generating comonad $\mathbb{P} = (P, \epsilon^P, \Delta^P)$. A *Nakayama functor* relative to \mathbb{P} is a functor $\nu: \mathcal{A} \rightarrow \mathcal{A}$ with an adjunction $(\nu, \nu^-, \theta, \lambda, \sigma): \mathcal{A} \rightarrow \mathcal{A}$ satisfying:

- (1) $\nu \circ P$ is right adjoint to P ;
- (2) $\lambda_P: P \rightarrow \nu^- \circ \nu \circ P$ is an isomorphism.

We also say that \mathbb{P} has a Nakayama functor ν . In Theorem 4.3.9 we show that a Nakayama functor is unique if it exists.

Let $I = \nu \circ P$ and let $\mathbb{I} = (I, \eta^I, \mu^I)$ denote the induced monad such that \mathbb{I} is right adjoint to \mathbb{P} . Let $\text{proj}_\mathbb{P}(\mathcal{A})$ denote the subcategory of \mathbb{P} -projective objects, and let $\text{inj}_\mathbb{I}(\mathcal{A})$ denote the subcategory of \mathbb{I} -injective objects. We have the following result.

Lemma 4.1.2. *Let \mathcal{A} be an abelian category with a generating comonad $P = (P, \epsilon^P, \Delta^P)$ and a Nakayama functor relative to P . The following holds:*

- (i) P is right adjoint to $P \circ \nu$;
- (ii) $I \circ \nu^-$ is right adjoint to I ;
- (iii) P accommodates Gorenstein objects;
- (iv) I accommodates Gorenstein objects;
- (v) $\sigma_I: \nu \circ \nu^- \circ I \rightarrow I$ is an isomorphism. In particular, the restriction $\nu: \text{proj}_P(\mathcal{A}) \rightarrow \text{inj}_I(\mathcal{A})$ is an equivalence with quasi-inverse $\nu^-: \text{inj}_I(\mathcal{A}) \rightarrow \text{proj}_P(\mathcal{A})$.

Proof. By axiom (2) for comonads with Nakayama functor we have an isomorphism $P \cong \nu^- \circ \nu \circ P$. Part (i) then follows since $\nu^- \circ (\nu \circ P)$ is right adjoint to $P \circ \nu$ by axiom (1). Also, since I is right adjoint to P we get that $I \circ \nu^-$ is right adjoint to $\nu \circ P = I$. This shows part (ii). Since P has a left and a right adjoint and $P \circ \nu \circ P = P \circ I$ is exact, we get that P accommodates Gorenstein objects.

For part (iv) note first that I is cogenerating by Lemma 2.4.12. Since I has a left and a right adjoint and the composition $I \circ \nu^- \circ I \cong I \circ P$ is exact, it follows that I accommodates Gorenstein objects, which proves (iv).

For part (v), recall that we have an equality $\sigma_{\nu \circ P} \circ \nu(\lambda_P) = 1$ from the triangle identities of the adjunction. Since λ_P is an isomorphism, it follows that $\sigma_{\nu \circ P} = \sigma_I$ is an isomorphism, which proves (v). \square

Note that the compositions $\nu \circ P$ and $\nu^- \circ I$ are exact, and hence we can apply Lemma 2.4.8 to ν and the dual to ν^- .

Let \mathcal{C} be a small category. We say that \mathcal{C} is *locally bounded* if for any object $c \in \mathcal{C}$ there are only finitely many objects in \mathcal{C} mapping nontrivially in and out of c . This means that for each $c \in \mathcal{C}$ we have

$$\begin{aligned} \mathcal{C}(c, c') &\neq 0 \quad \text{for only finitely many } c' \in \mathcal{C} \\ \mathcal{C}(c', c) &\neq 0 \quad \text{for only finitely many } c' \in \mathcal{C}. \end{aligned}$$

Following the conventions in [DSS] we say that \mathcal{C} is *Hom-finite* if $\mathcal{C}(c, c') \in \text{proj } k$ for all $c, c' \in \mathcal{C}$.

In the next example we consider the module category $\text{Mod } \mathcal{C}$ of a category \mathcal{C} . Background on such categories can be found in [M].

Example 4.1.3. Let k be a commutative ring, and let \mathcal{C} be a small, k -linear, locally bounded, and Hom-finite category. Let $k(\text{ob } \mathcal{C})$ be the category with the same objects as \mathcal{C} , and with morphisms

$$k(\text{ob } \mathcal{C})(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 \neq c_2, \\ k & \text{if } c_1 = c_2. \end{cases}$$

An object in $k(\text{ob } \mathcal{C})\text{-Mod}$ is just a tuple $(V^c)_{c \in \mathcal{C}}$ of k -modules, indexed over the objects of \mathcal{C} . Let $i: k(\text{ob } \mathcal{C}) \rightarrow \mathcal{C}$ be the inclusion. The restriction functor

$$i^*: \mathcal{C}\text{-Mod} \rightarrow k(\text{ob } \mathcal{C})\text{-Mod}, \quad i^*(F) = (F(c))_{c \in \mathcal{C}}$$

has a left adjoint

$$i_! : k(\text{ob } \mathcal{C})\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}, \quad i_!((V^c)_{c \in \mathcal{C}}) = \bigoplus_{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes_k V^c$$

and a right adjoint

$$i_* : k(\text{ob } \mathcal{C})\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}, \quad i_*((V^c)_{c \in \mathcal{C}}) = \bigoplus_{c \in \mathcal{C}} \text{Hom}_k(\mathcal{C}(-, c), V^c).$$

Let $\mathbf{P} = (P, \epsilon, \Delta)$ be the induced comonad on $\mathcal{C}\text{-Mod}$, where

$$P := i_! \circ i^*, \quad P(F) = \bigoplus_{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes_k F(c).$$

Let $D := \text{Hom}_k(-, k) : \text{Mod-}k \rightarrow \text{Mod-}k$ be the dual, and let $D\mathcal{C}$ be the \mathcal{C} -bimodule given by $D\mathcal{C}(c_1, c_2) = D(\mathcal{C}(c_2, c_1))$. The functor

$$\nu := D\mathcal{C} \otimes_{\mathcal{C}} - : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod} \quad \nu(F)(c) = D(\mathcal{C}(c, -)) \otimes_{\mathcal{C}} F$$

has a right adjoint

$$\begin{aligned} \nu^- &:= \text{Hom}_{\mathcal{C}\text{-Mod}}(D\mathcal{C}, -) : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod} \\ \nu^-(F)(c) &= \text{Hom}_{\mathcal{C}\text{-Mod}}(D(\mathcal{C}(-, c)), F). \end{aligned}$$

We want to show that ν is a Nakayama functor relative to \mathbf{P} . To this end, note that there are natural isomorphisms

$$\begin{aligned} \nu \circ P(F) &= D\mathcal{C} \otimes_{\mathcal{C}} \left(\bigoplus_{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes_k F(c) \right) \\ &\cong \bigoplus_{c \in \mathcal{C}} ((D\mathcal{C} \otimes_{\mathcal{C}} \mathcal{C}(c, -)) \otimes_k F(c)) \cong \bigoplus_{c \in \mathcal{C}} D(\mathcal{C}(-, c)) \otimes_k F(c) \\ &\cong \bigoplus_{c \in \mathcal{C}} \text{Hom}_k(\mathcal{C}(-, c), F(c)) \end{aligned}$$

where the last isomorphism follows since the map

$$g : D(V) \otimes_k W \rightarrow \text{Hom}_k(V, W) \tag{4.1.4}$$

given by $g(f \otimes w)(v) = f(v)w$ is an isomorphism for k -modules V, W when V is finitely generated projective. Also, we have that

$$\begin{aligned} \nu^- \circ \nu \circ P(F) &= \text{Hom}_{\mathcal{C}\text{-Mod}}(D\mathcal{C}, \bigoplus_{c \in \mathcal{C}} \text{Hom}_k(\mathcal{C}(-, c), F(c))) \\ &= \text{Hom}_{\mathcal{C}\text{-Mod}}(D\mathcal{C}, \prod_{c \in \mathcal{C}} \text{Hom}_k(\mathcal{C}(-, c), F(c))) \\ &\cong \prod_{c \in \mathcal{C}} \text{Hom}_{\mathcal{C}\text{-Mod}}(D\mathcal{C}, \text{Hom}_k(\mathcal{C}(-, c), F(c))) \\ &= \bigoplus_{c \in \mathcal{C}} \text{Hom}_{\mathcal{C}\text{-Mod}}(D\mathcal{C}, \text{Hom}_k(\mathcal{C}(-, c), F(c))) \end{aligned}$$

since \mathcal{C} is locally bounded and limits and colimits in $\mathcal{C}\text{-Mod}$ are taken point-wise. Finally, we have isomorphisms

$$\begin{aligned} \nu^- \circ \nu \circ P(F) &\cong \bigoplus_{c \in \mathcal{C}} \text{Hom}_{\mathcal{C}\text{-Mod}}(D\mathcal{C}, \text{Hom}_k(\mathcal{C}(-, c), F(c))) \\ &\cong \bigoplus_{c \in \mathcal{C}} \text{Hom}_k(\mathcal{C}(-, c) \otimes_{\mathcal{C}} D\mathcal{C}, F(c)) \\ &\cong \bigoplus_{c \in \mathcal{C}} \text{Hom}_k(D(\mathcal{C}(c, -)), F(c)) \cong \bigoplus_{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes_k F(c) = P(F) \end{aligned}$$

where the last isomorphism follows from (4.1.4) and the fact that $D^2\mathcal{C} \cong \mathcal{C}$ since \mathcal{C} is Hom-finite. This gives an inverse to the unit $\lambda_P \circ P \rightarrow \nu^- \circ \nu \circ P$, and therefore ν is a Nakayama functor relative to P .

Remark 4.1.5. Locally bounded Hom-finite categories are one of the main object of study [DSS]. In [DSS, Theorem 4.6] they assume that \mathcal{C} has a Serre functor relative to k . In our language this implies that the comonad P on $\mathcal{C}\text{-Mod}$ is 0-Gorenstein, see Theorem 5.1.7.

The example above can be generalized to functor categories $\mathcal{A}^{\mathcal{C}}$ where \mathcal{A} is any abelian category and \mathcal{C} is a small, k -linear, Hom-finite, and locally bounded category. We will do this in more detail in an upcoming paper.

Example 4.1.6. Let k be a commutative ring, and let Λ be a k -algebra which is finitely generated and projective as a k -module. This is a special case of Example 4.1.3 where $\mathcal{C} = \Lambda$ has only one object. It follows that

$$P = (\Lambda \otimes_k -) \circ \text{res}_k^\Lambda : \Lambda\text{-Mod} \rightarrow \Lambda\text{-Mod}$$

gives rise to a comonad P on $\Lambda\text{-Mod}$ with Nakayama functor

$$\nu = D\Lambda \otimes_\Lambda - : \Lambda\text{-Mod} \rightarrow \Lambda\text{-Mod}$$

where $\text{res}_k^\Lambda : \Lambda\text{-Mod} \rightarrow \text{Mod-}k$ denotes the restriction functor.

4.2. Basic properties. Let P be a generating comonad on \mathcal{A} , and assume it has a Nakayama functor ν . We let $T := P \circ \nu$ and $S := I \circ \nu^-$ denote the left adjoint of P and right adjoint of I .

Lemma 4.2.1. *Let $A \in \mathcal{A}$ and $i \in \mathbb{N}$ be arbitrary. The following holds:*

- (i) $H_i(\nu; A) = 0$ if and only if $H_i(T; A) = 0$;
- (ii) $H^i(\nu^-; A) = 0$ if and only if $H^i(S; A) = 0$.

Proof. By Lemma 2.4.8 part (ii) and dimension shifting it is sufficient to show this for $i = 1$. Let $0 \rightarrow A' \xrightarrow{s} Q \xrightarrow{t} A \rightarrow 0$ be an exact sequence in \mathcal{A} with Q being P -projective. If $H_1(\nu; A) = 0$, then the sequence $0 \rightarrow \nu(A') \xrightarrow{\nu(s)} \nu(Q) \xrightarrow{\nu(t)} \nu(A) \rightarrow 0$ is exact. Applying P then gives an exact sequence $0 \rightarrow T(A') \xrightarrow{T(s)} T(Q) \xrightarrow{T(t)} T(A) \rightarrow 0$, which shows that $H_1(T; A) = 0$. Conversely, assume $H_1(T; A) = 0$. The sequence $0 \rightarrow P \circ \nu(A') \xrightarrow{P \circ \nu(s)} P \circ \nu(Q) \xrightarrow{P \circ \nu(t)} P \circ \nu(A) \rightarrow 0$ is then exact. Since P

is faithful, it follows that $0 \rightarrow \nu(A') \xrightarrow{\nu(s)} \nu(Q) \xrightarrow{\nu(t)} \nu(A) \rightarrow 0$ is exact, and hence $H_1(\nu; A) = 0$. This proves (i). Statement (ii) is proved dually. \square

Hence, Gorenstein P-flat objects can also be defined using ν -exact sequences.

Proposition 4.2.2. *Let $A \in \mathcal{A}$ be arbitrary. The following holds:*

- (i) *If $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$, then $\nu(A) \in \mathcal{G}_I \text{ inj}(\mathcal{A})$;*
- (ii) *If $A \in \mathcal{G}_I \text{ inj}(\mathcal{A})$, then $\nu^-(A) \in \mathcal{G}_P \text{ flat}(\mathcal{A})$;*
- (iii) *If $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$, then $\lambda_A: A \rightarrow \nu^- \circ \nu(A)$ is an isomorphism;*
- (iv) *If $A \in \mathcal{G}_I \text{ inj}(\mathcal{A})$, then $\sigma_A: \nu \circ \nu^-(A) \rightarrow A$ is an isomorphism.*

In particular, the restriction $\nu: \mathcal{G}_P \text{ flat}(\mathcal{A}) \rightarrow \mathcal{G}_I \text{ inj}(\mathcal{A})$ is an equivalence with quasi-inverse $\nu^-: \mathcal{G}_I \text{ inj}(\mathcal{A}) \rightarrow \mathcal{G}_P \text{ flat}(\mathcal{A})$.

Proof. Let $Q_\bullet = \cdots Q_{-1} \xrightarrow{s_1} Q_0 \xrightarrow{s_0} Q_1 \xrightarrow{s_1} \cdots$ be a T -exact sequence with P-projective components. Applying ν gives an exact sequence

$$\nu(Q_\bullet) = \cdots \xrightarrow{\nu(s_{-2})} \nu(Q_{-1}) \xrightarrow{\nu(s_{-1})} \nu(Q_0) \xrightarrow{\nu(s_0)} \nu(Q_1) \xrightarrow{\nu(s_1)} \cdots$$

by Lemma 4.2.1. Applying ν^- and using Lemma 4.1.2 part (v) we get an isomorphism $\nu^- \circ \nu(Q_\bullet) \cong Q_\bullet$. Hence, by Lemma 4.2.1 $\nu(Q_\bullet)$ is S -exact. Since ν sends P-projective objects to I-injective objects, the complex $\nu(Q_\bullet)$ has I-injective components. Hence, if $Z^0(Q_\bullet) = A$, then $Z^0(\nu(Q_\bullet)) = \nu(A) \in \mathcal{G}_I \text{ inj}(\mathcal{A})$. This shows (i). Now consider the exact sequence $0 \rightarrow \nu(A) \rightarrow \nu(Q_0) \xrightarrow{\nu(s_1)} \nu(Q_1)$. Applying ν^- gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & Q_0 & \xrightarrow{s_1} & Q_1 \\ & & \downarrow \lambda_A & & \downarrow \lambda_{Q_0} & & \downarrow \lambda_{Q_1} \\ 0 & \longrightarrow & \nu^- \circ \nu(A) & \longrightarrow & \nu^- \circ \nu(Q_0) & \xrightarrow{\nu^- \circ \nu(s_1)} & \nu^- \circ \nu(Q_1) \end{array}$$

where the lower row is exact since ν^- is left exact. Hence, since λ_{Q_0} and λ_{Q_1} are isomorphisms, it follows that λ_A is an isomorphism. This proves part (iii) of the proposition. Part (ii) and (iv) are proved dually. \square

4.3. Uniqueness of Nakayama functor. In this section we show that the Nakayama functor associated to a generating comonad is unique if it exists. We assume throughout this subsection that the comonad P on \mathcal{A} is generating. We also fix the notation

$$\begin{aligned} (T, P, \phi^{T \dashv P}, \alpha^{T \dashv P}, \beta^{T \dashv P}): \mathcal{A} &\rightarrow \mathcal{A} \\ (P, I, \phi^{P \dashv I}, \alpha^{P \dashv I}, \beta^{P \dashv I}): \mathcal{A} &\rightarrow \mathcal{A} \\ (I, S, \phi^{I \dashv S}, \alpha^{I \dashv S}, \beta^{I \dashv S}): \mathcal{A} &\rightarrow \mathcal{A} \end{aligned}$$

for the adjunctions if they exist.

Lemma 4.3.1. *Assume P has a Nakayama functor ν . The following holds:*

$$\begin{aligned} (i) \quad & \alpha^{I \dashv S} = I(\lambda_P) \circ \alpha^{P \dashv I}; \\ (ii) \quad & \beta^{I \dashv S} = \sigma \circ \nu(\beta_{\nu^-}^{P \dashv I}). \end{aligned}$$

Proof. We have the identity

$$\phi^{I \dashv S}: \mathcal{A}(I, -) = \mathcal{A}(\nu \circ P, -) \xrightarrow{\theta} \mathcal{A}(P, \nu^-) \xrightarrow{\phi^{P \dashv I}} \mathcal{A}(-, I \circ \nu^-) = \mathcal{A}(-, S).$$

It follows that

$$\alpha^{I \dashv S} = \phi^{I \dashv S}(1_I) = \phi^{P \dashv I} \circ \theta(1_{\nu \circ P}) = \phi^{P \dashv I}(\lambda_P) = I(\lambda_P) \circ \alpha^{P \dashv I}$$

and

$$\begin{aligned} \beta^{I \dashv S} &= (\phi^{I \dashv S})^{-1}(1_S) = \theta^{-1} \circ (\phi^{P \dashv I})^{-1}(1_{I \circ \nu^-}) = \theta^{-1}(\beta_{\nu^-}^{P \dashv I}) \\ &= \sigma \circ \nu(\beta_{\nu^-}^{P \dashv I}). \end{aligned}$$

□

Proposition 4.3.2. *Assume \mathbf{P} has a Nakayama functor ν . The map*

$$I(\lambda_P): IP \rightarrow I\nu^- \nu P = SI$$

induces an isomorphism of monads

$$I(\lambda_P): (IP, \alpha^{P \dashv I}, I(\beta_P^{P \dashv I})) \xrightarrow{\cong} (SI, \alpha^{I \dashv S}, S(\beta_I^{I \dashv S})).$$

Proof. The map $I(\lambda_P)$ is an isomorphism by (2) of 4.1.1. Hence, we only need to show that $I(\lambda_P)$ is a morphism of monads. Note first that $I(\lambda_P) \circ \alpha^{P \dashv I} = \alpha^{I \dashv S}$ by Lemma 4.3.1 (i). It therefore only remains to show that the diagram

$$\begin{array}{ccccc} IP & \xrightarrow{I(\lambda_{PIP})} & SIIP & \xrightarrow{SII(\lambda_P)} & SISI \\ I(\beta_P^{P \dashv I}) \downarrow & & & & \downarrow S(\beta_I^{I \dashv S}) \\ IP & \xrightarrow{I(\lambda_P)} & & & SI \end{array}$$

commutes. By Lemma 4.3.1 part (ii) we have $\beta^{I \dashv S} = \sigma \circ \nu(\beta_{\nu^-}^{P \dashv I})$. Hence,

$$\begin{aligned} S(\beta_I^{I \dashv S}) \circ SII(\lambda_P) \circ I(\lambda_{PIP}) &= S(\sigma_I) \circ S\nu(\beta_{\nu^-}^{P \dashv I}) \circ SII(\lambda_P) \circ I(\lambda_{PIP}) \\ &= S(\sigma_I) \circ S\nu(\beta_{\nu^-}^{P \dashv I} \circ PI(\lambda_P)) \circ I(\lambda_{PIP}) \\ &= S(\sigma_I) \circ S\nu(\lambda_P) \circ S\nu(\beta_P^{P \dashv I}) \circ I(\lambda_{PIP}) \\ &= S(\sigma_I \circ \nu(\lambda_P)) \circ I(\nu^- \nu(\beta_P^{P \dashv I})) \circ \lambda_{PIP} \\ &= I(\nu^- \nu(\beta_P^{P \dashv I})) \circ \lambda_{PIP} \end{aligned}$$

by naturality, where the last equality follows from the triangle identities. Since

$$I(\nu^- \nu(\beta_P^{P \dashv I})) \circ \lambda_{PIP} = I(\lambda_P \circ \beta_P^{P \dashv I}) = I(\lambda_P) \circ I(\beta_P^{P \dashv I})$$

by naturality, the claim follows. □

We now show the converse of Proposition 4.3.2; if $P = (P, \epsilon^P, \Delta^P)$ is a generating comonad and there exist adjunctions $P \dashv I \dashv S$ and a natural isomorphism $\gamma: (IP, \alpha^{P \dashv I}, I(\beta_P^{P \dashv I})) \xrightarrow{\cong} (SI, \alpha^{I \dashv S}, S(\beta_I^{I \dashv S}))$ of monads then P has a Nakayama functor.

Lemma 4.3.3. *Let P be as above. Then there is an equivalence*

$$\nu'_{\text{can}}: \overline{\text{im}}P \rightarrow \overline{\text{im}}I$$

acting as identity on objects, and sending a morphism $f: P(X) \rightarrow P(Y)$ to

$$\nu'_{\text{can}}(f) := \phi^{I \dashv S}(\gamma \circ \phi^{P \dashv I}(f)): I(X) \rightarrow I(Y).$$

Proof. By Lemma 2.2.5 we have equivalences $\overline{\text{im}}P \cong \text{Kl}((IP, \alpha^{P \dashv I}, I(\beta_P^{P \dashv I})))$ and $\overline{\text{im}}I \cong \text{Kl}((SI, \alpha^{I \dashv S}, S(\beta_I^{I \dashv S})))$. Since γ is an isomorphism of monads it induces an equivalence $\text{Kl}((IP, \alpha^{P \dashv I}, I(\beta_P^{P \dashv I}))) \cong \text{Kl}((SI, \alpha^{I \dashv S}, S(\beta_I^{I \dashv S})))$. It is easy to see that ν'_{can} is the composite of these functors. \square

In terms of the unit and counit we have

$$\nu'_{\text{can}}(f) := \beta_{I(Y)}^{I \dashv S} \circ I(\gamma_Y) \circ II(f) \circ I(\alpha_X^{P \dashv I}): I(X) \rightarrow I(Y). \quad (4.3.4)$$

We let $\nu'_{\text{can}}: \overline{\text{im}}I \rightarrow \overline{\text{im}}P$ denote the inverse of ν'_{can} .

Lemma 4.3.5. *Let P be as above and let $f: X \rightarrow Y$ be a morphism in \mathcal{A} . Then $\nu'_{\text{can}}(P(f)) = I(f)$.*

Proof. Note that

$$\begin{aligned} \nu'_{\text{can}}(P(f)) &= \beta_{I(Y)}^{I \dashv S} \circ I(\gamma_Y) \circ IIP(f) \circ I(\alpha_X^{P \dashv I}) \\ &= \beta_{I(Y)}^{I \dashv S} \circ ISI(f) \circ I(\gamma_X) \circ I(\alpha_X^{P \dashv I}) \\ &= I(f) \circ \beta_{I(X)}^{I \dashv S} \circ I(\gamma_X) \circ I(\alpha_X^{P \dashv I}) \end{aligned}$$

by naturality of γ and $\beta^{I \dashv S}$. Since γ is a morphism of monads, we have that $\gamma_X \circ \alpha_X^{P \dashv I} = \alpha_X^{I \dashv S}$. Hence

$$I(f) \circ \beta_{I(X)}^{I \dashv S} \circ I(\gamma_X) \circ I(\alpha_X^{P \dashv I}) = I(f) \circ \beta_{I(X)}^{I \dashv S} \circ I(\alpha_X^{I \dashv S}) = I(f)$$

where the last equality follows from the triangle identities of the adjunction. This proves the claim. \square

For an object $A \in \mathcal{A}$, let

$$\begin{aligned} \text{mor}_P(A) &:= i \circ \epsilon_{\text{Ker } \epsilon_A^P}^P: P(\text{Ker } \epsilon_A^P) \rightarrow P(A) \\ \text{mor}_I(A) &:= \eta_{\text{Coker } \eta_A^I}^I \circ p: I(A) \rightarrow I(\text{Coker } \eta_A^I) \end{aligned}$$

denote the compositions, where $i: \text{Ker } \epsilon_A^P \rightarrow P(A)$ is the inclusion and $p: I(A) \rightarrow \text{Coker } \eta_A^I$ is the projection. These induce functors

$$\begin{aligned} \text{mor}_P: \mathcal{A} &\rightarrow \text{Mor}(\overline{\text{im}}P) & A &\rightarrow \text{mor}_P(A) \\ \text{mor}_I: \mathcal{A} &\rightarrow \text{Mor}(\overline{\text{im}}I) & A &\rightarrow \text{mor}_I(A) \end{aligned}$$

where $\text{Mor}(\overline{\text{im}}P)$ (resp $\text{Mor}(\overline{\text{im}}I)$) is the category of morphisms in $\overline{\text{im}}P$ (resp $\overline{\text{im}}I$). The functors ν'_{can} and ν'^{-}_{can} gives equivalences

$$\begin{aligned}\nu'_{\text{can}} &: \text{Mor}(\overline{\text{im}}P) \rightarrow \text{Mor}(\overline{\text{im}}I) \\ \nu'^{-}_{\text{can}} &: \text{Mor}(\overline{\text{im}}I) \rightarrow \text{Mor}(\overline{\text{im}}P)\end{aligned}$$

defined pointwise. Now consider the functors

$$\begin{aligned}\nu_{\text{can}} &:= \text{Coker} \circ \nu'_{\text{can}} \circ \text{mor}_P: \mathcal{A} \rightarrow \mathcal{A} \\ \nu_{\text{can}}^{-} &:= \text{Ker} \circ \nu'^{-}_{\text{can}} \circ \text{mor}_I: \mathcal{A} \rightarrow \mathcal{A}\end{aligned}$$

where $\text{Ker}: \text{Mor}(\overline{\text{im}}I) \rightarrow \mathcal{A}$ and $\text{Coker}: \text{Mor}(\overline{\text{im}}P) \rightarrow \mathcal{A}$ are the kernel and cokernel functors.

Lemma 4.3.6. *Let P be as above. We have an adjunction*

$$(\nu_{\text{can}}, \nu_{\text{can}}^{-}, \theta_{\text{can}}, \lambda_{\text{can}}, \sigma_{\text{can}}): \mathcal{A} \rightarrow \mathcal{A}.$$

Proof. Let $f: \nu_{\text{can}}(A_1) \rightarrow A_2$ be a morphism in \mathcal{A} . By definition, we have an exact sequence

$$I(\text{Ker } \epsilon_{A_1}^P) \xrightarrow{\nu'_{\text{can}}(\text{mor}_P(A_1))} I(A_1) \rightarrow \nu_{\text{can}}(A_1) \rightarrow 0$$

Consider the composite

$$\overline{f} := I(A_1) \rightarrow \nu_{\text{can}}(A_1) \xrightarrow{f} A_2 \xrightarrow{\eta_{A_2}^!} I(A_2)$$

It satisfies $\overline{f} \circ \nu'_{\text{can}}(\text{mor}_P(A_1)) = 0$ and $\text{mor}_I(A_2) \circ \overline{f} = 0$. Applying ν'^{-}_{can} to \overline{f} gives a morphism $\nu'^{-}_{\text{can}}(\overline{f}): P(A_1) \rightarrow P(A_2)$ satisfying $\nu'^{-}_{\text{can}}(\overline{f}) \circ \text{mor}_P(A_1) = 0$ and $\nu'^{-}_{\text{can}}(\text{mor}_I(A_2)) \circ \nu'^{-}_{\text{can}}(\overline{f}) = 0$. Since we have exact sequences

$$P(\text{Ker } \epsilon_{A_1}^P) \xrightarrow{\text{mor}_P(A_1)} P(A_1) \xrightarrow{\epsilon_{A_1}^P} A_1 \rightarrow 0$$

and

$$0 \rightarrow \nu_{\text{can}}^{-}(A_2) \rightarrow P(A_2) \xrightarrow{\nu'^{-}_{\text{can}}(\text{mor}_I(A_2))} P(\text{Coker } \eta_{A_2}^!)$$

it follows that the morphism $\nu'^{-}_{\text{can}}(\overline{f})$ induces a morphism $\theta_{\text{can}}(f): A_1 \rightarrow \nu_{\text{can}}^{-}(A_2)$. Obviously, the map $f \rightarrow \theta_{\text{can}}(f)$ is bijective. Now let $g: A_2 \rightarrow A_3$ be a morphism in \mathcal{A} . By naturality of $\eta^!$ it follows that

$$\overline{g \circ f} = I(g) \circ \overline{f}: I(A_1) \rightarrow I(A_3).$$

Applying ν'^{-}_{can} to this gives

$$\nu'^{-}_{\text{can}}(\overline{g \circ f}) = P(g) \circ \nu'^{-}_{\text{can}}(f): P(A_1) \rightarrow P(A_3)$$

by Lemma 4.3.5. Since we have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \nu_{\text{can}}^{-}(A_2) & \longrightarrow & P(A_2) \xrightarrow{\nu'^{-}_{\text{can}}(\text{mor}_I(A_2))} P(\text{Coker } \eta_{A_2}^!) \\ & & \downarrow \nu_{\text{can}}^{-}(g) & & \downarrow P(g) \\ 0 & \longrightarrow & \nu_{\text{can}}^{-}(A_3) & \longrightarrow & P(A_3) \xrightarrow{\nu'^{-}_{\text{can}}(\text{mor}_I(A_3))} P(\text{Coker } \eta_{A_3}^!) \end{array}$$

with exact rows, it follows that $\theta_{\text{can}}(g \circ f) = \nu_{\text{can}}^-(g) \circ \theta_{\text{can}}(f)$. Similarly, one can show that $\theta_{\text{can}}(f \circ \nu_{\text{can}}(h)) = \theta_{\text{can}}(f) \circ h$. Hence, θ_{can} is natural, and therefore we get the required adjunction. \square

Lemma 4.3.7. *Let \mathbf{P} be as above. There exist natural isomorphisms $\nu_{\text{can}} \circ P \cong I$ and $\nu_{\text{can}}^- \circ I \cong P$.*

Proof. The maps

$$\begin{aligned} \epsilon_{P(A)}^{\mathbf{P}}: P(P(A)) &\rightarrow P(A) \\ \text{mor}_{\mathbf{P}}(P(A)): P(\text{Ker } \epsilon_{P(A)}^{\mathbf{P}}) &\rightarrow \text{im mor}_{\mathbf{P}}(P(A)) \end{aligned}$$

are split epimorphism for any object $A \in \mathcal{A}$. Hence, applying ν'_{can} to the exact sequence

$$P(\text{Ker } \epsilon_{P(A)}^{\mathbf{P}}) \xrightarrow{\text{mor}_{\mathbf{P}}(P(A))} P(P(A)) \xrightarrow{\epsilon_{P(A)}^{\mathbf{P}}} P(A) \rightarrow 0$$

gives an exact sequence

$$I(\text{Ker } \epsilon_{P(A)}^{\mathbf{P}}) \xrightarrow{\nu'_{\text{can}}(\text{mor}_{\mathbf{P}}(P(A)))} I(P(A)) \rightarrow I(A) \rightarrow 0.$$

Since A was arbitrary we get a natural isomorphism $\nu_{\text{can}} \circ P \cong I$. Similarly, one can show that $\nu_{\text{can}}^- \circ I \cong P$. \square

Proposition 4.3.8. *Let \mathbf{P} be as above. Then ν_{can} is a Nakayama functor.*

Proof. It follows from Lemma 4.3.7 that $\nu_{\text{can}} \circ P$ is right adjoint to P . Furthermore, by Lemma 4.3.7 there exists an equivalence $P \cong \nu_{\text{can}}^- \circ \nu_{\text{can}} \circ P$. This implies that the unit $\lambda_P: P \rightarrow \nu^- \circ \nu \circ P$ is an isomorphism itself. Hence, ν_{can} is a Nakayama functor. \square

Theorem 4.3.9. *Let $\mathbf{P} = (P, \epsilon^{\mathbf{P}}, \Delta^{\mathbf{P}})$ be a generating comonad on \mathcal{A} . The following statements are equivalent:*

- (i) \mathbf{P} has a Nakayama functor ν ;
- (ii) There are adjunctions $P \dashv I \dashv S$ and a natural isomorphism

$$\gamma: (IP, \alpha^{P \dashv I}, I(\beta_P^{P \dashv I})) \xrightarrow{\cong} (SI, \alpha^{I \dashv S}, S(\beta_I^{I \dashv S}))$$

of monads.

Furthermore, in this case we have a natural isomorphism $\nu \cong \nu_{\text{can}}$.

Proof. The equivalence (i) \iff (ii) follows from Proposition 4.3.2 and Proposition 4.3.8.

To prove the existence of a natural isomorphism $\nu \cong \nu_{\text{can}}$, it is sufficient to show that $\nu_{\text{can}}(f) = \nu(f): I(A_1) \rightarrow I(A_2)$ for a morphism $f: P(A_1) \rightarrow P(A_2)$. From the identities $I\nu^- = S$ and $\gamma = I(\lambda_P)$, and from the naturality of λ and $\beta^{I \dashv S}$ we get that

$$\begin{aligned} \nu_{\text{can}}(f) &= \beta_{I(A_2)}^{I \dashv S} \circ II(\lambda_{P(A_2)}) \circ II(f) \circ I(\alpha_{A_1}^{P \dashv I}) \\ &= \beta_{I(A_2)}^{I \dashv S} \circ II(\nu^- \nu(f)) \circ II(\lambda_{P(A_1)}) \circ I(\alpha_{A_1}^{P \dashv I}) \\ &= \nu(f) \circ \beta_{I(A_1)}^{I \dashv S} \circ II(\lambda_{P(A_1)}) \circ I(\alpha_{A_1}^{P \dashv I}). \end{aligned}$$

By Lemma 4.3.1 part (ii) we know that $\beta^{I \dashv S} = \sigma \circ \nu(\beta_{\nu^-}^{P \dashv I})$. Hence, it follows that

$$\begin{aligned} \nu_{\text{can}}(f) &= \nu(f) \circ \sigma_{I(A_1)} \circ \nu(\beta_{P(A_1)}^{P \dashv I} \circ PI(\lambda_{P(A_1)})) \circ I(\alpha_{A_1}^{P \dashv I}) \\ &= \nu(f) \circ \sigma_{I(A_1)} \circ \nu(\lambda_{P(A_1)}) \circ \nu(\beta_{P(A_1)}^{P \dashv I}) \circ I(\alpha_{A_1}^{P \dashv I}) \\ &= \nu(f) \circ \nu(\beta_{P(A_1)}^{P \dashv I} \circ P(\alpha_{A_1}^{P \dashv I})) = \nu(f) \end{aligned}$$

using the naturality of $\beta^{P \dashv I}$, the triangle identity for σ and λ , and the triangle identity for $\beta^{P \dashv I}$ and $\alpha^{P \dashv I}$. The claim follows. \square

5. GORENSTEIN CATEGORIES FOR COMONADS

5.1. Main Result. We make the following assumption for this subsection.

Setting 5.1.1. Let $P = (P, \epsilon^P, \Delta^P)$ be a generating comonad with Nakayama functor $\nu: \mathcal{A} \rightarrow \mathcal{A}$ relative to P . We let $(\nu, \nu^-, \theta, \lambda, \sigma): \mathcal{A} \rightarrow \mathcal{A}$ denote the adjunction, $\mathbf{l} = (I, \eta^{\mathbf{l}}, \mu^{\mathbf{l}})$ the right adjoint monad to P with $I = \nu \circ P$, $\mathbf{T} = (T, \eta^{\mathbf{T}}, \mu^{\mathbf{T}})$ the left adjoint monad to P with $T = P \circ \nu$, and $S = (S, \epsilon^S, \Delta^S)$ the right adjoint comonad to \mathbf{l} .

Definition 5.1.2. The comonad P is called *Gorenstein* if there exists an $n \in \mathbb{N}$ such that $H_i(\nu; A) = 0$ and $H^i(\nu^-; A) = 0$ for all $A \in \mathcal{A}$ and $i \geq n$.

The following result gives a simpler description of $\mathcal{G}_P \text{ flat}(\mathcal{A})$ and $\mathcal{G}_{\mathbf{l}} \text{ inj}(\mathcal{A})$ when P is Gorenstein.

Theorem 5.1.3. *Assume P is Gorenstein. The following holds:*

- (i) $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$ if and only if $H_i(\nu; A) = 0$ for all $i \in \mathbb{N}$;
- (ii) $A \in \mathcal{G}_{\mathbf{l}} \text{ inj}(\mathcal{A})$ if and only if $H^i(\nu^-; A) = 0$ for all $i \in \mathbb{N}$.

Proof. If $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$, then $H_i(A; T) = 0$ for all $i \in \mathbb{N}$, and hence it follows that $H_i(A; \nu) = 0$ for all $i \in \mathbb{N}$ by Lemma 4.2.1. For the converse, note that if $H_i(A; \nu) = 0$ for all $i \in \mathbb{N}$, then $A \in T\text{-Acy}$ by Lemma 4.2.1. Hence, by Lemma 3.2.14 we only need to show that $A \in \Omega_P^\infty(\mathcal{A})$. To this end, choose an exact sequence

$$\cdots \xrightarrow{s_{-3}} Q_{-2} \xrightarrow{s_{-2}} Q_{-1} \xrightarrow{s_{-1}} A \rightarrow 0$$

with Q_i being P -projective. Applying ν gives an exact sequence

$$\cdots \xrightarrow{\nu(s_{-3})} \nu(Q_{-2}) \xrightarrow{\nu(s_{-2})} \nu(Q_{-1}) \xrightarrow{\nu(s_{-1})} \nu(A) \rightarrow 0$$

since $H_i(A; \nu) = 0$ for all $i \in \mathbb{N}$. Also, since $\nu(Q_i)$ is \mathbf{l} -injective and $H^i(A'; \nu^-) = 0$ for all $A' \in \mathcal{A}$ and $i \geq n$, it follows by Lemma 2.4.8 and dimension shifting that $H^i(\nu(A); \nu^-) = 0$ and $H^i(\text{Ker } \nu(s_j); \nu^-) = 0$ for all

$i \in \mathbb{N}$ and j . Therefore, we have a commutative diagram

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{s_{-3}} & Q_{-2} & \xrightarrow{s_{-2}} & Q_{-1} & \xrightarrow{s_{-1}} & A \longrightarrow 0 \\
 & & \downarrow \lambda_{Q_{-2}} & & \downarrow \lambda_{Q_{-1}} & & \downarrow \lambda_A \\
 \cdots & \xrightarrow{\nu^-\nu(s_{-3})} & \nu^-\nu(Q_{-2}) & \xrightarrow{\nu^-\nu(s_{-2})} & \nu^-\nu(Q_{-1}) & \xrightarrow{\nu^-\nu(s_{-1})} & \nu^-\nu(A) \longrightarrow 0
 \end{array}$$

where the rows are exact. Hence, the morphism $\lambda_A: A \rightarrow \nu^-\nu(A)$ is an isomorphism. Finally, choose an exact sequence

$$0 \rightarrow \nu(A) \rightarrow J_1 \xrightarrow{t_1} J_2 \xrightarrow{t_2} \cdots$$

with J_i being l-injective. Applying ν^- and using that λ_A is an isomorphism gives us an exact sequence

$$0 \rightarrow A \rightarrow \nu^-(J_1) \xrightarrow{\nu^-(t_1)} \nu^-(J_2) \xrightarrow{\nu^-(t_2)} \cdots$$

with $\nu^-(J_i)$ being P-projective. Since $H_i(A'; \nu) = 0$ for all $i \geq n$ and $A' \in \mathcal{A}$ it follows that this sequence is ν -exact. Therefore, by Lemma 4.2.1 the sequence is T -exact, and so $A \in \Omega_{\mathcal{P}}^\infty(\mathcal{A})$. This proves part (i). Part (ii) is proved dually. \square

It follows from Theorem 5.1.3 that if \mathcal{P} is Gorenstein, then

$$\text{res. dim}_{\mathcal{G}_{\mathcal{P}} \text{ flat}(\mathcal{A})}(\mathcal{A}) < \infty \quad \text{and} \quad \text{cores. dim}_{\mathcal{G}_1 \text{ inj}(\mathcal{A})}(\mathcal{A}) < \infty.$$

Our goal now is to prove the analogue of the equality in part 2) of [EEG, Theorem 2.28], i.e that when \mathcal{P} is Gorenstein the following numbers are equal:

- 1) $\text{res. dim}_{\mathcal{G}_{\mathcal{P}} \text{ flat}(\mathcal{A})}(\mathcal{A})$;
- 2) $\text{cores. dim}_{\mathcal{G}_1 \text{ inj}(\mathcal{A})}(\mathcal{A})$;
- 3) The smallest integer n_1 such that $H_i(\nu; A) = 0$ for all $i \geq n_1$ and $A \in \mathcal{A}$;
- 4) The smallest integer n_2 such that $H^i(\nu^-; A) = 0$ for all $i \geq n_2$ and $A \in \mathcal{A}$.

In order to prove this we need some preparation. We let $\text{im } \nu$ (resp $\text{im } \nu^-$) denote the subcategory of \mathcal{A} consisting of objects A such $A \cong \nu(A')$ ($A \cong \nu^-(A')$) for some object $A' \in \mathcal{A}$.

Lemma 5.1.4. *Let $A \in \mathcal{A}$. The following holds:*

- (i) $A \in \text{im } \nu$ if and only if there exists an exact sequence $J_0 \rightarrow J_1 \rightarrow A \rightarrow 0$ with J_0, J_1 being l-injective;
- (ii) $A \in \text{im } \nu^-$ if and only if there exists an exact sequence $0 \rightarrow A \rightarrow Q_0 \rightarrow Q_1$ with Q_0, Q_1 being P-projective.

Proof. For any object $A' \in \mathcal{A}$ choose an exact sequence $Q_0 \rightarrow Q_1 \rightarrow A' \rightarrow 0$ with Q_0 and Q_1 being P-projective. By applying ν and using that it is right exact and sends P-projective objects to l-injective objects, we get one

direction of part (i). For the converse, assume we have an exact sequence $J_0 \xrightarrow{s} J_1 \rightarrow A \rightarrow 0$ with J_0, J_1 being l-injective. Since $\sigma_{J_i}: \nu \circ \nu^-(J_i) \rightarrow J_i$ is an isomorphism, it follows that

$$A = \text{Coker } s \cong \text{Coker } \nu \circ \nu^-(s) \cong \nu(\text{Coker } \nu^-(s)).$$

This proves part (i). Part (ii) is proved dually. \square

By Proposition 2.2.9, the functor $\text{Ker } \epsilon^P: \mathcal{A} \rightarrow \mathcal{A}$ is right adjoint to the functor $\text{Coker } \eta^T: \mathcal{A} \rightarrow \mathcal{A}$. Hence, for $m \in \mathbb{N}$ the functor $(\text{Ker } \epsilon^P)^m \circ \nu^-$ is right adjoint to $\nu \circ (\text{Coker } \eta^T)^m$.

Lemma 5.1.5. *Let $A \in \mathcal{A}$ and $m \in \mathbb{N}$. The following holds:*

- (i) *If $A = \nu \circ (\text{Coker } \eta^T)^m(A')$ for an object $A' \in \mathcal{A}$, then there exists an exact sequence*

$$J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{m+1} \rightarrow A \rightarrow 0$$

with J_i being l-injective;

- (ii) *If $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$, then $\nu \circ (\text{Coker } \eta^T)^m(A) \in \mathcal{G}_l \text{ inj}(\mathcal{A})$;*
 (iii) *If $A \in \mathcal{G}_l \text{ inj}(\mathcal{A})$, then $(\text{Ker } \epsilon^P)^m \circ \nu^-(A) \in \mathcal{G}_P \text{ flat}(\mathcal{A})$.*

Proof. We prove (i). Consider the sequence

$$\begin{aligned} A' \xrightarrow{\eta_{A'}^T} T(A') \xrightarrow{s_0} T(\text{Coker } \eta^T(A')) \xrightarrow{s_1} \cdots \xrightarrow{s_{m-1}} T((\text{Coker } \eta^T)^{m-1}(A')) \\ \xrightarrow{p_{m-1}} (\text{Coker } \eta^T)^m(A') \rightarrow 0 \end{aligned} \quad (5.1.6)$$

where p_i is the canonical projection and s_i is the composite

$$\begin{aligned} T((\text{Coker } \eta^T)^i(A')) \\ \xrightarrow{p_i} (\text{Coker } \eta^T)^{i+1}(A') \xrightarrow{\eta_{(\text{Coker } \eta^T)^{i+1}(A')}^T} T((\text{Coker } \eta^T)^{i+1}(A')). \end{aligned}$$

Since $T(\eta^T): T \rightarrow T \circ T$ is a monomorphism, $T = P \circ \nu$, and P is faithful, it follows that $\nu(\eta^T): \nu \rightarrow \nu \circ T$ is a monomorphism. Hence, applying ν to (5.1.6) gives an exact sequence

$$0 \rightarrow \nu(A') \xrightarrow{\nu(\eta_{A'}^T)} J_2 \xrightarrow{\nu(s_0)} J_3 \xrightarrow{\nu(s_1)} \cdots \xrightarrow{\nu(s_{m-1})} J_{m+1} \xrightarrow{\nu(p_{m-1})} A \rightarrow 0$$

where $\nu \circ T((\text{Coker } \eta^T)^i(A')) = J_{i+2}$. Part (i) now follows by Lemma 5.1.4 and the fact that ν sends P-projective to l-injective objects.

Part (ii) and (iii) follow from the fact that $\text{Ker } \epsilon^P$ and $\text{Coker } \eta^T$ preserve objects in $\mathcal{G}_P \text{ flat}(\mathcal{A})$. \square

We now prove the main result of this section.

Theorem 5.1.7. *The following are equivalent:*

- (a) *P is Gorenstein;*
 (b) *$\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) < \infty$;*
 (c) *$\text{cores. dim}_{\mathcal{G}_l \text{ inj}(\mathcal{A})}(\mathcal{A}) < \infty$.*

Moreover, if this holds, then the following numbers coincide:

- (i) $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A})$;
- (ii) $\text{cores. dim}_{\mathcal{G}_I \text{ inj}(\mathcal{A})}(\mathcal{A})$;
- (iii) The smallest integer n_1 such that $H_i(\nu; A) = 0$ for all $i \geq n_1$ and $A \in \mathcal{A}$;
- (iv) The smallest integer n_2 such that $H^i(\nu^-; A) = 0$ for all $i \geq n_2$ and $A \in \mathcal{A}$.

If this common number is n , we say that P is n -Gorenstein.

Proof. The implications (a) \implies (b) and (a) \implies (c) follows from Theorem 5.1.3. Assume there exists an integer $n \geq 2$ such that $\text{cores. dim}_{\mathcal{G}_I \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq n$. Our goal is to show that $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq n$. To this end, let $A \in \mathcal{A}$ and consider the exact sequence

$$0 \rightarrow (\text{Ker } \epsilon^P)^n(A) \xrightarrow{i_n} P((\text{Ker } \epsilon^P)^{n-1}(A)) \xrightarrow{s_{n-1}} \dots \\ \dots \xrightarrow{s_3} P((\text{Ker } \epsilon^P)^2(A)) \xrightarrow{s_2} P((\text{Ker } \epsilon^P)^1(A)) \xrightarrow{s_1} P(A) \xrightarrow{\epsilon_A^P} A \rightarrow 0$$

where s_j is the composition

$$P((\text{Ker } \epsilon^P)^j(A)) \xrightarrow{\epsilon_{(\text{Ker } \epsilon^P)^j(A)}^P} (\text{Ker } \epsilon^P)^j(A) \xrightarrow{i_j} P((\text{Ker } \epsilon^P)^{j-1}(A))$$

and i_j is the inclusion. By Lemma 5.1.4 part (ii) there exists an object $A' \in \mathcal{A}$ such that $(\text{Ker } \epsilon^P)^2(A) \cong \nu^-(A')$. This implies that

$$(\text{Ker } \epsilon^P)^n(A) \cong \text{Ker}(\epsilon^P)^{n-2}(\nu^-(A')).$$

For simplicity we write $R = (\epsilon^P)^{n-2} \circ \nu^-$ and $L = \nu \circ (\text{Coker } \eta^T)^{n-2}$. By Lemma 5.1.5 part (i) and our assumption we know that $L(A'') \in \mathcal{G}_I \text{ inj}(\mathcal{A})$ for all $A'' \in \mathcal{A}$. Hence, by Lemma 5.1.5 part (iii) it follows that

$$R \circ L \circ R(A') \in \mathcal{G}_P \text{ flat}(\mathcal{A}).$$

By the triangle identities for the adjunction between L and R , we get that $(\text{Ker } \epsilon^P)^n(A) \cong R(A')$ is a direct summand of $R \circ L \circ R(A')$. Hence, by Theorem 3.2.15 part (iv)

$$(\text{Ker } \epsilon^P)^n(A) \in \mathcal{G}_P \text{ flat}(\mathcal{A})$$

This shows that $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq n$.

Now assume $\text{cores. dim}_{\mathcal{G}_I \text{ inj}(\mathcal{A})}(\mathcal{A}) \leq 1$. By the argument above we know that $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq 2$. Let $A \in \mathcal{A}$ be arbitrary, and choose an exact sequence

$$0 \rightarrow \text{Ker } s \xrightarrow{i} Q_0 \xrightarrow{s} Q_1 \xrightarrow{p} A \rightarrow 0$$

with Q_0, Q_1 being P -projective. Since $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq 2$, we get that $\text{Ker } s \in \mathcal{G}_P \text{ flat}(\mathcal{A})$. Consider the exact sequence $0 \rightarrow \text{Ker } s \xrightarrow{i} Q_0 \xrightarrow{q} \text{im } s \rightarrow 0$. Applying ν to this gives an exact sequence

$$\nu(\text{Ker } s) \xrightarrow{\nu(i)} \nu(Q_0) \xrightarrow{\nu(q)} \nu(\text{im } s) \rightarrow 0.$$

Hence, we have an epimorphism $\nu(\text{Ker } s) \xrightarrow{p'} \text{Ker } \nu(q) \rightarrow 0$. It follows by Lemma 5.1.4 part (i) that $\nu(Q_0) \in \mathcal{G}_l \text{inj}(\mathcal{A})$, and hence $\text{Ker } \nu(q) \in \mathcal{G}_l \text{inj}(\mathcal{A})$ since $\text{cores. dim}_{\mathcal{G}_l \text{flat}(\mathcal{A})}(\mathcal{A}) \leq 1$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccc} \text{Ker } s & \xrightarrow{i} & Q_0 & \xrightarrow{s} & Q_1 \\ \downarrow \lambda_{\text{Ker } s} & & \downarrow \lambda_{Q_0} & & \downarrow \lambda_{Q_1} \\ \nu^- \nu(\text{Ker } s) & \xrightarrow{\nu^- \nu(i)} & \nu^- \nu(Q_0) & \xrightarrow{\nu^- \nu(s)} & \nu^- \nu(Q_1) \end{array}$$

The vertical morphisms are isomorphisms by Proposition 4.2.2 part (iii). Hence, the morphism $\nu^- \nu(\text{Ker } s) \xrightarrow{\nu^- \nu(i)} \nu^- \nu(Q_0)$ is the kernel of $\nu^- \nu(s)$. In particular, it is a monomorphism. On the other hand, $\nu^- \nu(i)$ is also equal to the composition

$$\nu^- \nu(\text{Ker } s) \xrightarrow{\nu^-(p')} \nu^-(\text{Ker } \nu(q)) \xrightarrow{\nu^-(j)} \nu^- \nu(Q_0)$$

where $j: \text{Ker } \nu(q) \rightarrow \nu(Q_0)$ is the inclusion. Since

$$\nu^- \nu(s) \circ \nu^-(j) = \nu^-(\nu(s) \circ j) = 0$$

and $\nu^-(j)$ is a monomorphism, it follows that $\nu^-(p')$ is an isomorphism. Now consider the commutative diagram

$$\begin{array}{ccc} \nu \nu^- \nu(\text{Ker } s) & \xrightarrow{\nu \nu^-(p')} & \nu \nu^-(\text{Ker } \nu(q)) \\ \downarrow \sigma_{\nu(\text{Ker } s)} & & \downarrow \sigma_{\text{Ker } \nu(q)} \\ \nu(\text{Ker } s) & \xrightarrow{p'} & \text{Ker } \nu(q) \end{array}$$

Since the vertical maps and the upper horizontal map are isomorphisms, it follows that p' is an isomorphism. Hence, the exact sequence $0 \rightarrow \text{Ker } s \xrightarrow{i} Q_0 \xrightarrow{q} \text{im } s \rightarrow 0$ is ν -exact, and therefore T -exact by Lemma 4.2.1 part (i). By Theorem 3.2.15 part (iii) it follows that $\text{im } s \in \mathcal{G}_P \text{flat}(\mathcal{A})$. This implies that $\text{res. dim}_{\mathcal{G}_P \text{flat}(\mathcal{A})}(\mathcal{A}) \leq 1$, and since A was arbitrary we get that $\text{res. dim}_{\mathcal{G}_P \text{flat}(\mathcal{A})}(\mathcal{A}) \leq 1$.

Finally, we consider the case when $\text{cores. dim}_{\mathcal{G}_l \text{inj}(\mathcal{A})}(\mathcal{A}) = 0$. This implies that ν^- is exact. Also, $\text{res. dim}_{\mathcal{G}_P \text{flat}(\mathcal{A})}(\mathcal{A}) \leq 1$ by the argument above. Let $A \in \mathcal{A}$ be arbitrary, and choose a right exact sequence

$$Q_0 \xrightarrow{s} Q_1 \xrightarrow{p} A \rightarrow 0$$

with Q_0, Q_1 being P -projective. Since ν^- is exact and ν is right exact, the sequence $\nu^- \nu(Q_0) \xrightarrow{\nu^- \nu(s)} \nu^- \nu(Q_1) \xrightarrow{\nu^- \nu(p)} \nu^- \nu(A) \rightarrow 0$ is exact. Hence we

have a commutative diagram

$$\begin{array}{ccccccc}
 Q_0 & \xrightarrow{s} & Q_1 & \xrightarrow{p} & A & \longrightarrow & 0 \\
 \downarrow \lambda_{Q_0} & & \downarrow \lambda_{Q_1} & & \downarrow \lambda_A & & \\
 \nu^- \nu(Q_0) & \xrightarrow{\nu^- \nu(s)} & \nu^- \nu(Q_1) & \xrightarrow{\nu^- \nu(p)} & \nu^- \nu(A) & \longrightarrow & 0
 \end{array}$$

with right exact rows. Since λ_{Q_0} and λ_{Q_1} are isomorphisms, it follows that λ_A is an isomorphism. Since $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq 1$, it follows by Lemma 5.1.4 part (ii) that $A \in \mathcal{G}_P \text{ flat}(\mathcal{A})$. Since A was arbitrary, we get that $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) = 0$.

The dual of the above argument shows that if $\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) \leq n$, then $\text{cores. dim}_{\mathcal{G}_I \text{ inj}(\mathcal{A})}(\mathcal{A}) \leq n$. Hence, it follows that

$$\text{res. dim}_{\mathcal{G}_P \text{ flat}(\mathcal{A})}(\mathcal{A}) = \text{cores. dim}_{\mathcal{G}_I \text{ inj}(\mathcal{A})}(\mathcal{A}).$$

Together with Theorem 5.1.3 this proves the claim. \square

5.2. Applications. We want to apply Theorem 5.1.7 to Example 4.1.3. In order to do this we need the following lemma.

Lemma 5.2.1. *Let \mathcal{C} be a small, k -linear, locally bounded, and Hom-finite category. Assume that $M \in \text{Mod } \mathcal{C}$ satisfy*

$$\begin{aligned}
 M(c) &\in \text{proj } k \quad \forall c \in \mathcal{C} \\
 M(c) &\neq 0 \text{ for only finitely many } c \in \mathcal{C}.
 \end{aligned}$$

Then there exists an exact sequence

$$\cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$$

where Q_i is a finitely generated projective right \mathcal{C} -module for all i .

Proof. Choose an epimorphism $p^c: k^{n_c} \rightarrow M(c) \rightarrow 0$ for each $c \in \mathcal{C}$ with $M(c) \neq 0$, where $n_c \in \mathbb{N}$. The composition

$$q^c: \mathcal{C}(-, c) \otimes_k k^{n_c} \xrightarrow{1 \otimes p^c} \mathcal{C}(-, c) \otimes_k M(c) \xrightarrow{g^c} M$$

is a morphism of right \mathcal{C} -modules, where $(g^c)_{c'}: \mathcal{C}(c', c) \otimes_k M(c) \xrightarrow{g^c} M(c')$ sends $f \otimes v$ to $M(f)(v)$. The induced map

$$\bigoplus_{c \in \mathcal{C}, M(c) \neq 0} \mathcal{C}(-, c) \otimes_k k^{n_c} \xrightarrow{\oplus q^c} M$$

is then an epimorphism. Let K be the kernel of this map. Then $K(c') \neq 0$ for only finitely many $c' \in \mathcal{C}$ since the same holds for $\bigoplus_{c \in \mathcal{C}, M(c) \neq 0} \mathcal{C}(-, c) \otimes_k k^{n_c}$. Also, $K(c')$ is the kernel of the epimorphism

$$\bigoplus_{c \in \mathcal{C}, M(c) \neq 0} \mathcal{C}(c', c) \otimes_k k^{n_c} \xrightarrow{\oplus q^c} M(c')$$

and since $M(c') \in \text{proj } k$ and $\bigoplus_{c \in \mathcal{C}, M(c) \neq 0} \mathcal{C}(c', c) \otimes_k k^{n_c} \in \text{proj } k$, we get that $K(c') \in \text{proj } k$. Hence, K satisfies the same properties as M . We can therefore repeat this construction, which proves the claim. \square

Let \mathcal{C} be a small, k -linear, locally bounded, and Hom-finite category. Recall from Example 4.1.3 that the functor

$$P: \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod} \quad P(F) = \bigoplus_{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes_k F(c)$$

gives rise to a comonad P on $\mathcal{C}\text{-Mod}$ with Nakayama functor

$$\nu = D\mathcal{C} \otimes_{\mathcal{C}} -: \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}.$$

Theorem 5.2.2. *Let k be a commutative ring, and let \mathcal{C} be a small, k -linear, locally bounded, and Hom-finite category. Assume*

$$\sup_{c \in \mathcal{C}} (\text{proj. dim } D(\mathcal{C}(-, c))) < \infty \quad \text{and} \quad \sup_{c \in \mathcal{C}} (\text{proj. dim } D(\mathcal{C}(c, -))) < \infty$$

as left and right \mathcal{C} -modules respectively. Then

$$\sup_{c \in \mathcal{C}} (\text{proj. dim } D(\mathcal{C}(-, c))) = \sup_{c \in \mathcal{C}} (\text{proj. dim } D(\mathcal{C}(c, -))).$$

Proof. By assumption the comonad P on $\mathcal{C}\text{-Mod}$ is Gorenstein. Hence, by Theorem 5.1.7 P is n -Gorenstein for some $n \in \mathbb{N}$. It follows that

$$\begin{aligned} \text{Ext}_{\mathcal{C}\text{-Mod}}^n(D\mathcal{C}, -) \neq 0 \quad \text{and} \quad \text{Ext}_{\mathcal{C}\text{-Mod}}^i(D\mathcal{C}, -) = 0 \text{ for } i > n \\ \text{Tor}_n^{\mathcal{C}}(D\mathcal{C}, -) \neq 0 \quad \text{and} \quad \text{Tor}_i^{\mathcal{C}}(D\mathcal{C}, -) = 0 \text{ for } i > n. \end{aligned}$$

We therefore get that

$$\sup_{c \in \mathcal{C}} (\text{proj. dim } D(\mathcal{C}(-, c))) = n = \sup_{c \in \mathcal{C}} (\text{flat. dim } D(\mathcal{C}(c, -))).$$

On the other hand, by Lemma 5.2.1 there exists an n th syzygy of $D(\mathcal{C}(c, -))$ which is finitely presented. Since finitely presented flat modules are projective, it follows that $\text{flat. dim } D(\mathcal{C}(c, -)) = \text{proj. dim } D(\mathcal{C}(c, -))$. This proves the claim. \square

Corollary 5.2.3. *Let k be a commutative ring, and let Λ be a k -algebra which is finitely generated and projective as a k -module. Assume that*

$$\text{proj. dim } D(\Lambda)_{\Lambda} < \infty \quad \text{and} \quad \text{proj. dim } {}_{\Lambda} D(\Lambda) < \infty.$$

Then we have that

$$\text{proj. dim } D(\Lambda)_{\Lambda} = \text{proj. dim } {}_{\Lambda} D(\Lambda).$$

Proof. This follows from Theorem 5.2.2 when $\mathcal{C} = \Lambda$ has one object. \square

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